

Local monomialization of a system of first integrals

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Abstract

Given an analytic singular foliation ω with n first integrals (f_1, \dots, f_n) such that $df_1 \wedge \dots \wedge df_n \neq 0$, we prove that there exists a local monomialization of the system of first integrals, i.e. there exist sequences of local blowings-up such that the strict transform of ω has n monomial first integrals $(\mathbf{x}^{\alpha_1}, \dots, \mathbf{x}^{\alpha_n})$, where $\mathbf{x}^{\alpha_i} = x_1^{\alpha_{i,1}} \dots x_m^{\alpha_{i,m}}$ and the set of multi-indexes $(\alpha_1, \dots, \alpha_n)$ is linearly independent.

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1 Introduction

To date, the existence of a reduction of singularities for general singular foliations is a difficult open problem. More precisely, given a couple (M, ω) where M is an algebraic or analytic variety and ω is a general singular foliation, a reduction of singularities of ω is a birational map $\sigma : \widetilde{M} \rightarrow M$ such that the strict transform of ω has only 'minimal' singularities (introduced in [Mc], following the approach of the Mori program), called Log-Canonical singularities.

In one hand, the problem is well-understood whenever the ambient variety has dimension smaller or equal than three (see [Ben, Se, P, McP, Ca] for a complete list of results). More precisely, the classical Bendixson-Seidenberg Theorem (see [Ben, Se]) gives a positive answer whenever the ambient dimension is two. For the three dimensional case, [P, McP] gives a positive answer for line foliations and [Ca] gives a positive answer for codimension one foliations. At the other hand, there is no result valid for dimension bigger or equal than four. In particular, there is not even a local result (such as a local uniformization) valid for dimension higher than three.

In what follows, we consider singular foliations with first integrals and we present a local reduction of the system of first integrals. In particular, if ω is a totally integrable singular foliation (e.g. ω is given by the level curves of an analytic map), Theorem 1.1 below gives rise to a local reduction of singularities for ω . More precisely, our main result is the following:

Theorem 1.1 (Main Theorem). *Let M be a non-singular analytic manifold, E be a SNC (simple normal crossing) divisor on M , p a point of M and ω a singular distribution with n analytic first integrals (f_1, \dots, f_n) over p such that $df_1 \wedge \dots \wedge df_n$ calculated at p is non-zero. Then, there exists a finite collection of morphisms $\Phi_i : (M_i, E_i) \rightarrow (M, E)$ such that:*

- I) The morphism Φ_i is a finite composition of adapted local blowings-ups (i.e local blowings-up with centers with SNC to E);*
- II) In each variety M_i , there exists a compact set $K_i \subset M_i$ such that the union of their images $\bigcup \Phi_i(K_i)$ is a compact neighborhood of p ;*
- III) The singular distribution ω_i , given by the strict transform (or analytic strict transform) of ω by Φ_i , is n -monomial integrable, i.e. for every point q_i in M_i , there exists an adapted coordinate system $\mathbf{x} = (x_1, \dots, x_m)$ centered at q (i.e. the exceptional divisor E_i is locally given by $\{x_1 \cdots x_l = 0\}$) and n first integrals $(\mathbf{x}^{\alpha_1}, \dots, \mathbf{x}^{\alpha_n})$ of ω_i . \mathcal{O}_q such that $\alpha_1 \wedge \dots \wedge \alpha_n \neq 0$.*

Notation 1.2 (Collection of local blowings-up). In order to simplify notation, a finite collection of morphisms $\Phi_i : (M_i, E_i) \rightarrow (M, E)$ that satisfies conditions [I] and [II] of Theorem 1.1 is called a *collection of local blowings-up*.

When ω is a totally integrable singular foliations, the class of singularities of the transforms ω_i is a sub-class of the Log-Canonical singularities that we call *monomial* singularities (see Definition 2.2 below). It comes as no surprise that monomial singularities are deeply related with *monomial mappings* (see [K, Cu1] for a definition). Indeed, the main idea to prove Theorem 1.1 is to

consider a local analytic mapping given by the first integrals (f_1, \dots, f_n) and to monomialize the level curves of this mapping, hence obtaining the final monomial form given in condition [III] of Theorem 1.1.

Before going into further details, let us remark that the problem of monomialization of maps is a very delicate problem, which does not follow from resolution of singular varieties. The best results, to date, are set in the algebraic category and are given by the series of articles [Cu1, Cu2, Cu3, ADK]. In particular, in [Cu1, Cu2], Cutkosky proves the existence of a local monomialization of maps using an algebraic construction that follows Zariski ideas for local uniformization of varieties (see [Z]). This result can be used to prove Theorem 1.1 in the algebraic category, but its proof does not seem to extend in an easy way for the analytic category (although it might be possible to do so using the *stars of Hironaka* - see [Cu4]).

We prove our main result using elementary geometrical arguments which are independent from [Cu1, Cu2]. The main idea is to adapt a global algorithm of monomialization of maps, valid only for low dimensions (see [Cu3]), for a higher dimensional case. In order to do that, we sacrifice the global property of the algorithm and we use techniques of [Bel] which allow one to preserve the monomiality of a singular foliation.

To exemplify the idea of the proof, let us consider the case where there are only three first integrals (f_1, f_2, f_3) . By resolution of singular varieties, we can assume that f_1 is a monomial function, i.e., $f_1 = \mathbf{x}^{\alpha_1}$. Now, we can adapt the algorithm given in [Cu3] in order to monomialize the second function f_2 and obtain a map $(f_1, f_2) = (\mathbf{x}^{\alpha_1}, \mathbf{x}^{\alpha_2})$ where α_1 and α_2 are linearly independent multi-indexes. A further simple adaptation of the algorithm given in [Cu3] would lead to a monomialization of the function f_3 which **breaks** the monomiality of the first pair (f_1, f_2) . Nevertheless, the techniques in [Bel, Bel] allow us to choose blowings-up which preserve the monomiality of an ambient singular foliation such as, in this case, $df_1 \wedge df_2$. This is exactly what we need in order to monomialize the level curves of the map (f_1, f_2, f_3) and prove Theorem 1.1.

The manuscript is divided in four sections counting the introduction. In the second section we introduce the main objects used in the rest of the manuscript, including the definitions and techniques of [Bel, Bel] that are going to be necessary. In the third section we introduce the notion of *foliated sub-ring sheaves* and we present the main idea of the proof: first, we give a sense for monomialization of a sub-ring sheaf (see Lemma 3.6 below); second, we prove Theorem 1.1 assuming the technical Theorem 3.7. The forth and last section is completely devoted to prove Theorem 3.7.

Finally, it is worth remarking that this manuscript is not completely self-contained: Theorems 2.11 and 2.12 below are proved in [Bel, Bel] and we do not reproduce their proofs in here.

2 Main Objects

2.1 Foliated Manifold

In what follows, a *foliated analytic manifold* is the triple (M, θ, E) where

- M is a smooth analytic manifold of dimension m over a field \mathbb{K} (where \mathbb{K} is \mathbb{R} or \mathbb{C});
- E is an ordered collection $E = (E^{(1)}, \dots, E^{(l)})$, where each $E^{(i)}$ is a smooth divisor on M such that $\sum_i E^{(i)}$ is a reduced divisor with simple normal crossings;
- θ is an involutive singular distribution defined over M and everywhere tangent to E .

We briefly recall the notion of singular distribution following closely [BaBo]. Let Der_M denote the sheaf of analytic vector fields over M , i.e. the sheaf of analytic sections of TM . An *involutive singular distribution* is a coherent sub-sheaf θ of Der_M such that for each point p in M , the stalk $\theta_p := \theta \cdot \mathcal{O}_p$ is closed under the Lie bracket operation.

Consider the quotient sheaf $Q = Der_M / \theta$. The *singular set* of θ is defined by the closed analytic subset $S = \{p \in M : Q_p \text{ is not a free } \mathcal{O}_p \text{ module}\}$. A singular distribution θ is called *regular* if $S = \emptyset$. On $M \setminus S$ there exists a unique analytic sub bundle L of $TM|_{M \setminus S}$ such that θ is the sheaf of analytic sections of L . We assume that the dimension of the \mathbb{K} vector space L_p is the same for all points p in $M \setminus S$ (this always holds if M is connected). It is called the *leaf dimension* of θ and denoted by d . In this case θ is called an *involutive d -singular distribution*.

2.2 Compact Notation

In what follows, it will be useful to have a compact notation for denoting a collection of monomials. To that end, let \mathbf{u} be a collection of k functions (u_1, \dots, u_k) and A be a $t \times k$ matrix:

$$A = \begin{bmatrix} \alpha_1 \\ \vdots \\ \alpha_t \end{bmatrix} = \begin{bmatrix} \alpha_{1,1} & \dots & \alpha_{1,k} \\ \vdots & \ddots & \vdots \\ \alpha_{t,1} & \dots & \alpha_{t,k} \end{bmatrix}$$

Then, we define:

$$\mathbf{u}^A = \begin{bmatrix} \mathbf{u}^{\alpha_1} \\ \vdots \\ \mathbf{u}^{\alpha_t} \end{bmatrix} = \begin{bmatrix} u_1^{\alpha_{1,1}} \dots u_k^{\alpha_{1,k}} \\ \vdots \\ u_1^{\alpha_{t,1}} \dots u_k^{\alpha_{t,k}} \end{bmatrix}$$

Lemma 2.1. *Let $\mathbf{u} = (u_1, \dots, u_k)$ and $\mathbf{x} = (x_1, \dots, x_r)$ be two collections of functions such that $\mathbf{u} = \mathbf{x}^B$ for some $k \times r$ matrix B . Then, for any $t \times k$ matrix A , we have that $\mathbf{u}^A = \mathbf{x}^{AB}$.*

Proof. Indeed, let β_i be the line vectors of \mathbf{B} , i.e $\mathbf{B} = \begin{bmatrix} \beta_1 \\ \vdots \\ \beta_k \end{bmatrix}$. Notice that, by definition $u_i = \mathbf{x}^{\beta_i}$, which implies that:

$$\mathbf{u}^A = \begin{bmatrix} u_1^{\alpha_{1,1}} \dots u_k^{\alpha_{1,k}} \\ \vdots \\ u_1^{\alpha_{t,1}} \dots u_k^{\alpha_{t,k}} \end{bmatrix} = \begin{bmatrix} \mathbf{x}^{\alpha_{1,1}\beta_1} \dots \mathbf{x}^{\alpha_{1,k}\beta_k} \\ \vdots \\ \mathbf{x}^{\alpha_{t,1}\beta_1} \dots \mathbf{x}^{\alpha_{t,k}\beta_k} \end{bmatrix} = \begin{bmatrix} \prod_{i=1}^r x_i^{\sum_{j=1}^k \alpha_{1,j}\beta_{j,i}} \\ \vdots \\ \prod_{i=1}^r x_i^{\sum_{j=1}^k \alpha_{t,j}\beta_{j,i}} \end{bmatrix} = \mathbf{x}^C$$

where

$$C = \begin{bmatrix} \sum_{j=1}^k \alpha_{1,j}\beta_{j,1} & \dots & \sum_{j=1}^k \alpha_{1,j}\beta_{j,r} \\ \vdots & \ddots & \vdots \\ \sum_{j=1}^k \alpha_{t,j}\beta_{j,1} & \dots & \sum_{j=1}^k \alpha_{t,j}\beta_{j,r} \end{bmatrix}$$

which is clearly equal to \mathbf{AB} . \square

2.3 Monomial singular distribution

Definition 2.2 (Monomial singular distribution). Given a foliated manifold (M, θ, E) , we say that the singular distribution θ is *monomial* at a point p if there exists set of generators $\{\partial_1, \dots, \partial_d\}$ of $\theta \cdot \mathcal{O}_p$ and a coordinate system $(\mathbf{u}, \mathbf{w}) = (u_1, \dots, u_k, w_{k+1}, \dots, w_m)$ centered at p such that:

- i) The exceptional divisor E is locally equal to $\{u_1 \dots u_l = 0\}$ for some $l \leq k$;
- ii) The singular distribution θ is everywhere tangent to E , i.e., $\theta \subset \text{Der}_M(-\log E)$;
- iii) Apart from re-indexing, the vector-fields ∂_i are of the form:

$$\partial_i = \partial w_{m+1-i} \text{ for } i \text{ from } 1 \text{ to } m-k, \text{ and}$$

$$\partial_i = \sum_{j=1}^k \alpha_{i,j} u_j \partial u_j \text{ (where } \alpha_{i,j} \in \mathbb{Q} \text{) otherwise}$$

- iv) If $\omega \subset \text{Der}_M(-\log E)$ is a d -singular distribution such that $\theta \subset \omega$ then $\theta = \omega$.

In this case, we say that (\mathbf{u}, \mathbf{w}) is a *monomial coordinate system* and that $\{\partial_1, \dots, \partial_d\}$ is a *monomial basis* of $\theta \cdot \mathcal{O}_p$.

Remark 2.3 (Geometrical Interpretation of (iv)). Assuming conditions [i – iii] above, it is clear that Property [iv] implies that the codimension one part of the singularity set of θ is contained in E .

Notation 2.4 (An special monomial coordinate system). In what follows, we sometimes need to emphasis one of the coordinates in \mathbf{w} . To that end, we will denote by $(\mathbf{u}, v, \mathbf{w})$ a monomial coordinate system and the vector field ∂_v is contained in $\theta \cdot \mathcal{O}_p$.

Monomial singular distributions are deeply related with the existence of monomial first integrals:

Lemma 2.5 (Monomial First Integrals). *Given a foliated manifold (M, θ, E) , the singular distribution θ is monomial if, and only if, for any monomial coordinate system $(\mathbf{u}, \mathbf{w}) = (u_1, \dots, u_k, w_{k+1}, \dots, w_m)$ centered at p there exists $m - d$ monomials $\mathbf{u}^{\mathbf{B}} = (\mathbf{u}^{\beta_1}, \dots, \mathbf{u}^{\beta_{m-d}})$, where \mathbf{B} has maximal rank, such that*

$$\theta \cdot \mathcal{O}_p = \{\partial \in \text{Der}_p(-\log E); \partial(\mathbf{u}^{\beta_i}) \equiv 0 \text{ for all } i\}$$

In this case, we call $\mathbf{u}^{\mathbf{B}}$ a complete system of first integrals.

Proof. First, let us assume that θ is a monomial singular distribution and let us fix a point p in M and a monomial coordinate system (\mathbf{u}, \mathbf{w}) . It is clear that if f is a first integral of θ , then it can not depend on any coordinate w , since all the derivations ∂_{w_i} are contained in $\theta \cdot \mathcal{O}_p$. So, consider a monomial \mathbf{u}^{β} and let us remark that:

$$\begin{aligned} \partial_i(\mathbf{u}^{\beta}) &\equiv 0 \text{ if } i \leq m - k, \text{ and} \\ \partial_i(\mathbf{u}^{\beta}) &\equiv \mathbf{u}^{\beta} \sum_{j=1}^k \alpha_{i,j} \beta_j, \text{ otherwise} \end{aligned}$$

So, the monomial \mathbf{u}^{β} is a first integral of θ if, and only if:

$$\sum_{j=1}^k \alpha_{i,j} \beta_j = 0 \text{ for all } m - k < i \leq d \quad (2.1)$$

Thus, there exists a $m - d$ linear subspace L of \mathbb{Q}^k that contain all vector β satisfying the equations 2.1. In particular, we can choose a system of generators $\{\beta_1, \dots, \beta_{m-d}\}$ of L such that $\beta_i \in \mathbb{Z}^k$. So, the $m - d$ monomials $\mathbf{u}^{\mathbf{B}} = (\mathbf{u}^{\beta_1}, \dots, \mathbf{u}^{\beta_{m-d}})$ are first integrals of θ and:

$$\theta \cdot \mathcal{O}_p \subset \{\partial \in \text{Der}_p(-\log E); \partial(\mathbf{u}^{\beta_i}) \equiv 0 \text{ for all } i \leq m - d\}$$

By the maximal condition [iv], we conclude that both singular distributions are equal.

Now, let θ be the singular distribution over p given by

$$\{\partial \in \text{Der}_p(-\log E); \partial(\mathbf{u}^{\beta_i}) \equiv 0 \text{ for all } i \leq m - d\}$$

and let us prove that this is a monomial singular distribution. First, it is clear that the vector-fields $X_i = \partial w_{m+1-i}$ for $i \leq m - k$ are all contained in θ . So, consider a vector-field of the form $Y = \sum_{j=1}^k \alpha_j u_j \partial_{u_j}$ and let us notice that:

$$Y(\mathbf{u}^{\beta_i}) = \mathbf{u}^{\beta_i} \left(\sum_{j=1}^k \alpha_j \beta_{i,j} \right)$$

So, since Y is clearly tangent to E , it belongs to θ if and only if:

$$\sum_{j=1}^k \alpha_j \beta_{i,j} = 0 \text{ for all } i \quad (2.2)$$

Consider the $d + k - m$ linear subspace L of \mathbb{Q}^k that contain all vectors α satisfying the equations 2.2. In particular, for any fixed system of generators $\{\alpha_1, \dots, \alpha_{d+k-m}\}$, we define $Y_i = \sum_j \alpha_{i,j} u_j \partial_{u_j}$, which are vector-fields contained in θ . It rests to prove that there are no other vector-fields in θ . Indeed, let Z be any vector-field contained in θ :

$$Z = \sum_{j=1}^k A_j \partial_{u_j} + \sum_{j=k+1}^m B_j \partial_{w_j}$$

Now

$$Z(\mathbf{u}^{\beta_i}) \equiv 0 \iff \mathbf{u}^{\beta_i} \sum_j \beta_{i,j} \frac{A_j}{u_j} \equiv 0$$

Thus, for each $j \leq k$ we can write $A_j = u_j \tilde{A}_j$. Indeed, either u_j divides A_j (and the result is trivial), or the terms $\beta_{i,j}$ are zero for all i . In this case, notice that the vector-field ∂_{u_j} is such that

$$\partial_{u_j}(\mathbf{u}^{\beta_i}) \equiv 0 \text{ for all } i \leq m - d$$

So, the coordinate u_j either can be changed into a w coordinate, or it is an exceptional variable. In this case, the vector-field ∂_{u_j} is not tangent to E and A_i must be divisible by u_i in order for the vector-field Z to be tangent to E . Thus:

$$Z = \sum u_j \tilde{A}_j \partial_{u_j} + \sum_{j=k+1}^m B_j \partial_{w_j}$$

We now can take the Taylor expansion of $Z(\mathbf{u}^{\beta_i})$:

$$Z(\mathbf{u}^{\beta_i}) = \mathbf{u}^{\beta_i} \sum_{j=1}^k \beta_{i,j} \tilde{A}_j = \mathbf{u}^{\beta_i} \sum_{(\delta, \gamma)} \mathbf{u}^{\delta} \mathbf{w}^{\gamma} \sum_{j=1}^k \beta_{i,j} A_{j, \delta, \gamma}$$

which implies that $\sum_{j=1}^k \beta_{i,j} A_{j, \delta, \gamma} = 0$ and, by the definition of the vector-fields Y_i , there exists $C_{i, \delta, \gamma}$ such that:

$$A_{j, \delta, \gamma} = \sum_i C_{i, \delta, \gamma} \alpha_{i,j}$$

So, let $C_i = \sum_{(\delta, \gamma)} C_{i, \delta, \gamma} \mathbf{u}^{\delta} \mathbf{w}^{\gamma}$. It is clear that

$$Z = \sum C_i Y_i + \sum B_i X_i$$

and, thus θ is equal to the singular distribution generated by the vector-fields $\{X_i, Y_i\}$. Since θ clearly satisfies the maximal condition (iv), we conclude the Lemma. \square

We also prove that the monomiality is an open property:

Lemma 2.6 (Openness of monomiality). *The monomiality is an open condition i.e. if θ is monomial at p in M , then there exists an open neighborhood U of p such that θ is monomial at every point q in U .*

Proof. For a proof coming directly from the definition, see Lemma 2.2.1 in [Bel]. In here, we present a different proof (using Lemma 2.5) which is useful for the current manuscript.

Indeed, let us fix a monomial coordinate system $(\mathbf{u}, \mathbf{w}) = (u_1, \dots, u_k, w_{k+1}, \dots, w_m)$ centered at p which is defined in a neighborhood U of p . Since θ is monomial, by Lemma 2.5, there exists $m - d$ monomials $\mathbf{u}^{\mathbf{B}} = (\mathbf{u}^{\beta_1}, \dots, \mathbf{u}^{\beta_{m-d}})$, such that

$$\theta \cdot \mathcal{O}_p = \{X \in \text{Der}_p(-\log E); X(\mathbf{u}^{\beta_i}) \equiv 0 \text{ for all } i\}$$

So, fix a point q of U and let (δ, γ) be its coordinate in the coordinate system (\mathbf{u}, \mathbf{w}) . Apart from re-indexing, we can assume that $\delta = (0, \dots, 0, \delta_{t+1}, \dots, \delta_k)$ for some $t \leq k$ and let us consider the coordinate system $(\mathbf{x}, \mathbf{y}, \mathbf{v}) = (x_1, \dots, x_t, y_{t+1}, \dots, y_k, v_{k+1}, \dots, v_m)$ where

$$\begin{aligned} x_i &= u_i \\ y_i &= u_i - \delta_i \\ v_i &= w_i - \gamma_i \end{aligned}$$

which is a coordinate system centered at q . We can now write:

$$\mathbf{u}^{\mathbf{B}} = \mathbf{x}^{\mathbf{B}_1} (\mathbf{y} - \boldsymbol{\delta})^{\mathbf{B}_2}$$

where \mathbf{B}_1 is a $k \times t$ matrix and \mathbf{B}_2 is a $k \times (m - d - t)$ matrix such that

$$\mathbf{B} = [\mathbf{B}_1 \quad \mathbf{B}_2]$$

Furthermore, apart from re-ordering the lines of the matrix \mathbf{B} , we can further write:

$$\mathbf{B} = \begin{bmatrix} \mathbf{B}'_1 & \mathbf{B}'_2 \\ \mathbf{B}''_1 & \mathbf{B}''_2 \end{bmatrix}$$

where $\mathbf{B}_1 = \begin{bmatrix} \mathbf{B}'_1 \\ \mathbf{B}''_1 \end{bmatrix}$ and the rank of \mathbf{B}'_1 is maximal and equal to the rank of \mathbf{B}_1 .

So, there exists a change of coordinates $(\mathbf{x}_{(1)}, \mathbf{y}_{(1)}, \mathbf{v}_{(1)})$ such that:

$$\mathbf{u}^{\mathbf{B}} = \mathbf{x}_{(1)}^{\mathbf{C}_1} (\mathbf{y}_{(1)} - \boldsymbol{\delta})^{\mathbf{C}_2}$$

where:

$$\mathbf{C} = [\mathbf{C}_1 \quad \mathbf{C}_2] = \begin{bmatrix} \mathbf{C}'_1 & \mathbf{C}'_2 \\ \mathbf{C}''_1 & \mathbf{C}''_2 \end{bmatrix} = \begin{bmatrix} \mathbf{B}'_1 & 0 \\ \mathbf{B}''_1 & \boldsymbol{\Lambda} \end{bmatrix}$$

where $\boldsymbol{\Lambda}$ is a maximal rank matrix of rational numbers. This implies that the collection $(\mathbf{x}_{(1)}^{\mathbf{B}'_1}, \mathbf{x}_{(1)}^{\mathbf{B}''_1} (\mathbf{y}_{(1)} - \boldsymbol{\delta})^{\boldsymbol{\Lambda}})$ is a collection of first integrals of $\theta \cdot \mathcal{O}_q$. Since \mathbf{B}'_1 has rank equal to \mathbf{B}_1 , we conclude that:

$$(\mathbf{x}_{(1)}^{\mathbf{B}'_1}, (\mathbf{y}_{(1)} - \boldsymbol{\delta})^{\boldsymbol{\Lambda}})$$

is another collection of first integrals of $\theta \cdot \mathcal{O}_q$. Furthermore, since $\boldsymbol{\Lambda}$ is of maximal rank, there exists a coordinate system $(\mathbf{x}_{(2)}, \mathbf{y}_{(2)}, \mathbf{z}_{(2)}, \mathbf{v}_{(2)})$ where $\mathbf{x}_{(2)} = \mathbf{x}_{(1)}$ and $\mathbf{v}_{(2)} = \mathbf{v}_{(1)}$ such that:

$$(\mathbf{y}_{(1)} - \boldsymbol{\delta})^{\boldsymbol{\Lambda}} = \mathbf{y}_{(2)} - \boldsymbol{\delta}_{(2)}$$

which finally implies that the monomial functions

$$(\mathbf{x}_{(1)}^{B'_1}, \mathbf{y}_{(2)})$$

are first integrals of $\theta.\mathcal{O}_q$. By the analyticity of θ and Lemma 2.5, we conclude that the singular distribution $\theta.\mathcal{O}_q$ is monomial. Since q is an arbitrary point in U , we conclude that the monomiality property is open. \square

2.4 The analytic strict transform

Given an *admissible* blowing-up $\sigma : (\widetilde{M}, \widetilde{E}) \rightarrow (M, E)$ (i.e. the center of blowings-up \mathcal{C} has SNC with E) let F be the exceptional divisor of the blowing-up. Consider the sheaf of $\mathcal{O}_{\widetilde{M}}$ -modules $\mathcal{B}lDer_{\widetilde{M}} := \mathcal{O}(F) \otimes_{\mathcal{O}_{\widetilde{M}}} Der_{\widetilde{M}}$. Notice that the morphism $\sigma : (\widetilde{M}, \widetilde{E}) \rightarrow (M, E)$ gives rise to an application:

$$\sigma^* : Der_M \longrightarrow \mathcal{B}lDer_{\widetilde{M}}$$

Now, consider the coherent sub-sheaf $Der_{\widetilde{M}}(-\log \widetilde{E})$ of $Der_{\widetilde{M}}$ composed by all the derivations which leave the exceptional divisor \widetilde{E} invariant. We notice that there exists a natural imersion:

$$\zeta : Der_{\widetilde{M}}(-\log \widetilde{E}) \longrightarrow \mathcal{B}lDer_{\widetilde{M}}$$

Finally, the *analytic strict transform* $\widetilde{\theta}$ of θ is the singular distribution:

$$\widetilde{\theta} = \zeta^{-1} \sigma^*(\theta)$$

where $\zeta^{-1}(\omega)$ stands for the coherent sub-sheaf of $Der_{\widetilde{M}}(-\log \widetilde{E})$ generated by the pre-image of ω . For a relation between the analytic strict transform and the usual strict transform, see remark 2.9 below.

2.5 θ -Admissible Blowings-up

Given a foliated manifold (M, θ, E) , the *generalized k -Fitting operation* (for $k \leq d$) is a mapping $\Gamma_{\theta, k}$ that associates to each coherent ideal sheaf \mathcal{I} over M the ideal sheaf $\Gamma_{\theta, k}(\mathcal{I})$ whose stalk at each point p in M is given by:

$$\Gamma_{\theta, k}(\mathcal{I}).\mathcal{O}_p = \langle \{ \det[X_i(f_j)]_{i, j \leq k}; X_i \in \theta_p, f_j \in \mathcal{I}.\mathcal{O}_p \} \rangle$$

where $\langle S \rangle$ stands for the ideal generated by the subset $S \subset \mathcal{O}_p$.

Now, consider a regular analytic sub-manifold \mathcal{C} of M and the reduced ideal sheaf $\mathcal{I}_{\mathcal{C}}$ that generates \mathcal{C} . We say that \mathcal{C} is a *θ -admissible center* if:

- \mathcal{C} is a regular closed sub-variety that has SNC with E ;
- There exists $0 \leq d_0 \leq d$ such that the k -generalized Fitting-ideal $\Gamma_{\theta, k}(\mathcal{I}_{\mathcal{C}})$ is equal to the structural ideal \mathcal{O}_M for all $k \leq d_0$ and is contained in the ideal sheaf $\mathcal{I}_{\mathcal{C}}$ otherwise.

In particular, we say that a center is *θ -invariant* if the number d_0 above is equal to 0. A θ -admissible blowing-up is a blowing-up with θ -admissible center. The following Theorem enlightens the interest of θ -admissible blowings-up:

Theorem 2.7 (θ -admissible blowings-up). (See Theorem 4.1.1 of [Bel]) Let (M, θ, E) be a monomial d -foliated manifold and:

$$\tau : (\widetilde{M}, \widetilde{\theta}, \widetilde{E}) \rightarrow (M, \theta, E)$$

a θ -admissible blowing-up. Then $\widetilde{\theta}$ is monomial.

Remark 2.8 (Proof of Theorem). In [Bel], a singular distribution is called monomial if it satisfies Properties [i-iii] of the Definition 2.2. So, Theorem 4.1.1 proves that $\widetilde{\theta}$ also satisfies Properties [i-iii], but Claims nothing about Property [iv]. Nevertheless, by Remark 2.3, it is clear that if θ satisfies condition [iv], then $\widetilde{\theta}$ also satisfies condition [iv], which gives rise to the formulation used on this work.

Remark 2.9 (Strict Transform of a monomial singular distribution θ). As a consequence of Theorem 2.7, if θ is monomial and $\tau : (\widetilde{M}, \widetilde{\theta}, \widetilde{E}) \rightarrow (M, \theta, E)$ is θ -admissible, then the analytic strict transform $\widetilde{\theta}$ coincides with the intersection of the strict transform of θ with $Der_{\widetilde{M}}(-\widetilde{E})$. In particular, they are equal in the non-dicritical case (i.e. if the strict transform is tangent to \widetilde{E}).

Because of Theorem, 2.7, the θ -admissible blowings-up are going to be an essential tool in this work. So, let us present a couple of examples and an intuitive description of the Definition:

Example 1: If \mathcal{C} is an admissible and θ -invariant center (i.e if all leafs of θ that intersects \mathcal{C} are contained in \mathcal{C}) it is θ -admissible.

Example 2: If \mathcal{C} is an admissible and θ -totally transverse center (i.e all vector-fields in θ are transverse to \mathcal{C}) it is θ -admissible.

Example 3: Let $(M, \theta, E) = (\mathbb{C}^3, \{\partial_x, \partial_y\}, \emptyset)$ and $\mathcal{C} = \{x = 0\}$. Then \mathcal{C} is a θ -admissible center, but it is neither invariant nor totally transverse. Indeed, $\Gamma_{\theta,1}(\mathcal{I}_{\mathcal{C}}) = \mathcal{O}_M$ and $\Gamma_{\theta,2}(\mathcal{I}_{\mathcal{C}}) \subset \mathcal{I}_{\mathcal{C}}$.

Example 4: Let $(M, \theta, E) = (\mathbb{C}^3, \{\partial_x, \partial_y\}, \emptyset)$ and $\mathcal{C} = \{x^2 - z = 0\}$. Then \mathcal{C} is **not** a θ -admissible center. Indeed, $\Gamma_{\theta,1}(\mathcal{I}_{\mathcal{C}}) = (x, z)$.

Remark 2.10 (Intuition of the Definition). (See Proposition 4.3.1 of [Bel]) If a center \mathcal{C} is θ -admissible then, for each point p in \mathcal{C} , there exists two singular distributions germs θ_{inv} and θ_{tr} such that: θ_p is generated by $\{\theta_{inv}, \theta_{tr}\}$; \mathcal{C} is θ_{inv} -invariant; and \mathcal{C} is θ_{tr} -totally transverse.

2.6 Foliated Ideal sheaf

A *foliated ideal sheaf* is a quadruple $(M, \theta, \mathcal{I}, E)$ where \mathcal{I} is a coherent and everywhere non-zero ideal sheaf of \mathcal{O}_M . Given an adapted blowing-up $\tau : (\widetilde{M}, \widetilde{\theta}, \widetilde{E}) \rightarrow (M, \theta, E)$, we define the transform $\widetilde{\mathcal{I}}$ of \mathcal{I} as the total transform $\mathcal{I} \cdot \mathcal{O}_{\widetilde{M}}$. We now recall two important results about foliated ideal sheaves. The first concerns θ -invariant global resolution of singularities:

Theorem 2.11 (θ -Invariant resolution of Ideal). (See Theorem 4.1.1 of [Bel]) Let $(M, \theta, \mathcal{I}, E)$ be an analytic d -foliated ideal sheaf such that \mathcal{I} is θ -invariant, i.e. $\theta(\mathcal{I}) \subset \mathcal{I}$. Then, for each point p in M and relatively compact open neighborhood U of p , there exists a sequence of θ -invariant blowings-up (i.e. all blowings-up are θ -admissible)

$$\tau : (\tilde{U}, \tilde{\theta}, \tilde{\mathcal{I}}, \tilde{E}) \rightarrow (U, \theta, \mathcal{I}, E)$$

such that the ideal sheaf $\tilde{\mathcal{I}}$ is a principal ideal sheaf with support contained in E_i . In particular, if θ is monomial, then $\tilde{\theta}$ is monomial.

The second concerns a θ -admissible local resolution of singularities:

Theorem 2.12 (θ -Resolution of Ideal). (See Theorem 1.1 of [Bel]) Let $(M, \theta, \mathcal{I}, E)$ be an analytic d -foliated ideal sheaf. Then, for every point p in M , there exists a θ -admissible collection of local blowings-up (i.e. all local blowings-up are θ -admissible)

$$\tau_i : (M_i, \theta_i, \mathcal{I}_i, E_i) \rightarrow (M, \theta, \mathcal{I}, E)$$

such that the ideal sheaf \mathcal{I}_i is a principal ideal sheaf with support contained in E_i . In particular, if θ is monomial, then θ_i is monomial.

3 Strategy of Proof

3.1 Foliated sub-ring sheaf

A *foliated sub-ring sheaf* is a quadruple $(M, \theta, \mathcal{R}, E)$ where \mathcal{R} is a coherent and everywhere non-zero sub-ring sheaf of \mathcal{O}_M . Furthermore, we assume that at each point p in M there exists a finite system of generators (f_1, \dots, f_n) of $\mathcal{R} \cdot \mathcal{O}_p$ such that

$$df_1 \wedge \dots \wedge df_n \neq 0$$

and, if g is another function of $\mathcal{R} \cdot \mathcal{O}_p$, then

$$df_1 \wedge \dots \wedge df_n \wedge dg \equiv 0$$

Remark 3.1. Notice that the above condition does not follow from coherence. For example, the sub-ring (x^2, xy, y^2) is coherent but does not satisfy the above condition.

Remark 3.2. In the notation of Theorem 1.1, the sub-ring \mathcal{R} stands for the ring of first integrals of the initial singular distribution ω .

Remark 3.3. Since our goal is a local result, we work with a fixed system of generators of \mathcal{R} .

Finally, $(M, \theta, \mathcal{R}, E)$ is said to be *trivial* at a point p if all functions f in $\mathcal{R} \cdot \mathcal{O}_p$ are first integrals of θ , i.e. $X(f) \equiv 0$ for all X in $\theta \cdot \mathcal{O}_p$.

3.2 Main Invariant

Let $(M, \theta, \mathcal{R}, E)$ be a monomial foliated sub-ring sheaf and consider a system of generators (f_1, \dots, f_n) of \mathcal{R} and monomial coordinate systems (\mathbf{u}, \mathbf{w}) of p . Then, there exists a maximal multi-index δ such that:

$$f_i = g_i + \mathbf{u}^\delta T_i \quad (3.1)$$

where g_i are first integrals of θ and all monomials in the Taylor expansion of $\mathbf{u}^\delta T_i$ are *not* first integrals of θ .

Definition 3.4 (Main Invariant). In the above notations, consider the coordinate dependent function

$$\nu(p, \theta, \mathcal{R}, (\mathbf{u}, \mathbf{w})) := \min\{|\boldsymbol{\lambda}| : \partial_{\mathbf{w}}^{\boldsymbol{\lambda}} T_i \text{ is a unit}\}.$$

where we assume that $\partial_{\mathbf{w}}^{\boldsymbol{\lambda}}$ is the identity if \mathbf{w} is empty, and $\nu(p) = \infty$ if there are no $\boldsymbol{\lambda}$ such that $\partial_{\mathbf{w}}^{\boldsymbol{\lambda}} T_i$ is a unit. We define the *tangency order* of $(M, \theta, \mathcal{R}, E)$ by:

$$\nu(p, \theta, \mathcal{R}) := \min\{\nu(p, \theta, \mathcal{R}, (\mathbf{u}, \mathbf{w})) : \text{for all } (\mathbf{u}, \mathbf{w})\}.$$

When there is no risk of confusion, we simply denote $\nu(p, \theta, \mathcal{R})$ by $\nu(p)$.

Lemma 3.5. *The tangency order is independent of the set of generators of \mathcal{R} and is upper-semicontinuous.*

Proof. At every point p , let I be the ideal generated by all the functions in $\theta[\mathcal{R}].\mathcal{O}_U$, where U is a neighborhood of p . Taking U sufficiently small, we can assume that I is generated by $\theta[(f_1, \dots, f_n)]$, i.e by the set $\{X(f_i); X \in \theta\}$. In particular, this implies that I is independent of the choice of generators of \mathcal{R} and of coordinate systems of M . Now, consider the sequence of ideals:

$$\begin{aligned} I_1 &= I \\ I_{i+1} &= I_i + \theta[I_i] \end{aligned}$$

Then, by Noetherianity, there exists an integer μ such that:

$$I_\mu = I_{\mu+1}$$

Now, in the fixed (\mathbf{u}, \mathbf{w}) monomial coordinate system, let us divide in two cases:

Assume that $\nu < \infty$: In this case, we claim that $\mu = \nu$. Indeed, in one hand by the definition of μ and ν , we know that $I_\nu = (\mathbf{u}^\delta) = I_\mu$. On the other hand, the ideal $I_{\nu-1}$ does not contain \mathbf{u}^δ yet (or it would contradict the definition of ν). This implies that $\mu = \nu$ and, since μ is clearly an upper semi-continuous invariant of the pair (θ, \mathcal{I}) , we conclude that ν is upper semi-continuous.

Assume that $\nu = \infty$: In this case, it is clear that the invariant ν can only droop in a neighborhood of p (thus, satisfies the condition for semi-continuity) and we only need to prove that it is independent of the choice of generators of \mathcal{R} . To that end, we claim that at every monomial coordinate system, the ideal sheaf I_μ is *not* monomial (which is an intrinsic property of I). Indeed, if it were monomial, it would have to be equal to \mathbf{u}^γ for some γ . By another hand, this monomial would factor out from $f_i - g_i$ (since it divides it), which implies that $\gamma = \delta$. But this contradicts the assumption that there exists no derivation of T_i which is a unit. Since this property is intrinsic of the ideal I , we conclude the Lemma. \square

Lemma 3.6 (Invariant zero or one). *Let $(M, \theta, \mathcal{R}, E)$ be a monomial d -foliated sub-ring sheaf and p a point of M where the invariant $\nu(p, \theta, \mathcal{R})$ is 0 or 1. Then, for any system of generators (f_1, \dots, f_n) of \mathcal{R} , there exists an index i_0 and a monomial coordinate system (\mathbf{u}, \mathbf{w}) such that:*

$$f_{i_0} = g_{i_0} + \mathbf{u}^\beta w_1^\epsilon$$

where we recall that g_{i_0} is a first integral of θ and $\mathbf{u}^\beta w_1^\epsilon$ is not a first integral (see equation 3.1). Moreover, the monomial \mathbf{u}^β divides all functions $f_i - g_i$ and the constant ϵ is 0 or 1. In particular, the $(d-1)$ -foliated sub-ring sheaf $(M, \omega, \mathcal{R}, E)$ given by $\omega = \{X \in \theta; X(f_{i_0}) \equiv 0\}$ is monomial.

Proof. Fix a monomial system of coordinates (\mathbf{u}, \mathbf{w}) and recall that, by Lemma 2.5, there exists a complete system of first integrals $\mathbf{u}^B = (\mathbf{u}^{\beta_1}, \dots, \mathbf{u}^{\beta_{m-d}})$ of θ . We now consider the cases where $\nu(p)$ is zero and one separately:

First, assume that $\nu(p) = 0$ (this is the case when $\epsilon = 0$). Without loss of generality, we can assume that T_1 is a unit. By the definition of the functions T_i , the multi-index δ has to be linearly independent with all the multi-indexes β_i . Thus, apart from a change of coordinates (which preserves all monomials), we can assume that $T_1 = 1$. So, the singular distribution $\omega = \{X \in \theta; X(f_1) \equiv 0\}$ has a complete system of first integrals given by $(\mathbf{u}^B, \mathbf{u}^\beta)$ which implies that it is monomial.

Now, assume that $\nu(p) = 1$ (this is the case when $\epsilon = 1$). In this case, there exists a coordinate system $(\mathbf{u}, v, \mathbf{w})$ such that $\partial_v T_1$ is a unit. So, apart from a change of coordinates in the v coordinate, we can assume that $T_1 = v$. Thus, the singular distribution $\omega = \{X \in \theta; X(f_1) \equiv 0\}$ has a complete system of first integrals given by $(\mathbf{u}^B, \mathbf{u}^\beta v)$ which implies that it is monomial. \square

We now turn to the main technical result of the manuscript:

Theorem 3.7 (Main Theorem 2). *Let $(M, \theta, \mathcal{R}, E)$ be a non-trivial monomial foliated sub-ring sheaf. Then, for each point p in M , there exists a θ -admissible collection of local blowings-up $\tau_i : (M_i, \theta_i, \mathcal{R}_i, E_i) \rightarrow (M, \theta, \mathcal{R}, E)$ such that, for every point q_i in the pre-image of p , the invariant $d(q_i, \theta_i, \mathcal{R}_i)$ is zero or one.*

The proof of the above Theorem is given in section 4. In the next subsection we show how this results proves the main Theorem 1.1.

3.3 Proof of Theorem 1.1 (Assuming Theorem 3.7)

Fixed the point p , recall that there exists n first integrals (f_1, \dots, f_n) of ω such that

$$df_1 \wedge \dots \wedge df_n \neq 0$$

So, let us consider a monomial m -foliated sub-ring sheaf $(U, \theta_{(0)}, \mathcal{R}, E)$ where:

- U is a sufficiently small neighborhood of p so that the first integrals (f_1, \dots, f_n) are everywhere defined and $df_1 \wedge \dots \wedge df_n \neq 0$ everywhere;
- \mathcal{R} is the sub-ring generated by the first integrals (f_1, \dots, f_n) ;

- $\theta_{(0)}$ is the monomial singular distribution $Der_M(-\log E)$ i.e the sheaf of derivations of M tangent to E

In this case, let us notice that the singular distribution $\omega \cap Der_M(-\log E)$ is obviously contained in $\theta_{(0)}$. The proof follows a recursive argument:

Claim 3.8. *Let ω be a singular distribution with n first integrals (f_1, \dots, f_n) and $(M, \theta_{(k)}, \mathcal{R}, E)$ be a monomial $(m - k)$ -foliated sub-ring sheaf such that:*

- i) \mathcal{R} is the sub-ring generated by global first integrals (f_1, \dots, f_n) of ω ;*
- ii) Apart from re-indexing the functions (f_1, \dots, f_n) , the singular distribution $\theta_{(k)}$ is equal to $\{X \in Der_M(-\log E); X(f_i) \equiv 0 \text{ for all } i \leq k\}$. In particular $\omega \cap Der_M(-\log E) \subset \theta_{(k)}$.*

Then, if $k < n$, there exists a collection of $\theta_{(k)}$ -admissible local blowings-up:

$$\Phi_i : (M_i, \theta_{i(k)}, \mathcal{R}_i, E_i) \rightarrow (M, \theta_{(k)}, \mathcal{R}, E)$$

such that, for each point q_i in the pre-image of q , there exists a monomial $[m - (k + 1)]$ -foliated sub-ring sheaf $(M_i, \theta_{i(k+1)}, \mathcal{R}_i, E_i)$ that satisfies properties [i] and [ii] in respect to the strict transform ω_i of ω i.e:

- i) \mathcal{R}_i is the sub-ring generated by global first integrals $\tau_i^*(f_1, \dots, f_n) = (f_1^*, \dots, f_n^*)$ of ω_i ;*
- ii) Apart from re-indexing the functions (f_1^*, \dots, f_n^*) , the singular distribution $\theta_{i(k+1)}$ is equal to $\{X \in Der_{M_i}(-\log E_i); X(f_i^*) \equiv 0 \text{ for all } i \leq k + 1\}$. In particular $\omega \cap Der_{M_i}(-\log E_i) \subset \theta_{i(k+1)}$.*

Proof. Indeed, since $df_1 \wedge \dots \wedge df_n \neq 0$ and $k < n$, the monomial foliated sub-ring sheaf $(M, \theta_{(k)}, \mathcal{R}, E)$ is non-trivial. Thus, by Theorem 3.7 there exists a collection of $\theta_{(k)}$ -admissible local blowings-up:

$$\Phi_i : (M_i, \theta_{i(k)}, \mathcal{R}_i, E_i) \rightarrow (M, \theta_{(k)}, \mathcal{R}, E)$$

such that, for every point q_i in the pre-image of q , the invariant $\nu(q_i, \theta_{i(k)}, \mathcal{R}_i)$ is either zero or one. So, by Lemma 3.6 there exists a monomial $(m - k - 1)$ -foliated sub-ring sheaf $(M_i, \theta_{i(k+1)}(q_i), \mathcal{R}_i, E_i)$ where

$$\theta_{i(k+1)}(q_i) = \{X \in \theta_{i(k)}; X(f_{i_0}^*) \equiv 0\}$$

for some index $i_0 > k$ (because $\theta_{i(k+1)}(q_i)$ is not equal to $\theta_{i(k)}$). So, by the compacity of the pre-image of p and Lemma 2.6, after shrinking M_i if necessary, we can suppose that the singular distribution $\theta_{i(k)}(q_i)$ is monomial everywhere in M_i . Thus, it is independent of the point q_i and we can simply denote it by $\theta_{i(k)}$. Moreover, since $\theta_{i(k)}$ is monomial, by Remark 2.9 we conclude that:

$$\theta_{i(k)} = \{X \in Der_{M_i}(-\log E_i); X(f_i^*) \equiv 0 \text{ for all } i \leq k\}$$

So, apart from re-indexing, we conclude that:

$$\begin{aligned} \theta_{i(k+1)} &= \{X \in \theta_{i(k)}; X(f_{k+1}^*) \equiv 0\} \\ &= \{X \in Der_{M_i}(-\log E_i); X(f_i^*) \equiv 0 \text{ for all } i \leq k + 1\} \end{aligned}$$

which proves the Claim. \square

So, it is clear we can recursively use the Claim over $(U, \theta_{(0)}, \mathcal{R}, E)$ in order to get a collection of local blowings-up:

$$\Phi_i : (U_i, \mathcal{R}_i, E_i) \rightarrow (U, \mathcal{R}, E)$$

where, for each point q_i in the pre-image of p , there exists a trivial monomial $(m-n)$ -foliated sub-ring sheaf $(U_i, \theta_{(m-n)}, \mathcal{R}_i, E_i)$ such that:

$$\theta_{(m-n)} = \{X \in \text{Der}_{U_i}(-\log E); X(f_i^*) \equiv 0 \text{ for all } i \leq n\}$$

Notice that the strict transform (or analytic strict transform) ω_i of ω has first integrals in \mathcal{R}_i , which implies that $\omega_i \cap \text{Der}_{U_i}(-\log E_i) \subset \theta_{(m-n)}$. Now, by Lemma 2.5, given a point q_i in U_i there exists a monomial coordinate system (\mathbf{u}, \mathbf{w}) centered at q_i and n -monomial first integrals $\mathbf{u}^{\mathbf{B}} = (\mathbf{u}^{\beta_1}, \dots, \mathbf{u}^{\beta_n})$ of $\theta_{(m-n)} \cdot \mathcal{O}_{q_i}$ where \mathbf{B} is of maximal rank. Since $\omega_i \cap \text{Der}_{U_i}(-\log E_i) \subset \theta_{(m-n)}$, it is clear that the monomials $\mathbf{u}^{\mathbf{B}} = (\mathbf{u}^{\beta_1}, \dots, \mathbf{u}^{\beta_n})$ are also first integrals of ω_i , which finishes the proof.

4 Dropping the Invariant

4.1 Basic Normal Form

Lemma 4.1 (Dealing with infinite invariant). *Let $(M, \theta, \mathcal{R}, E)$ be an adapted monomial foliated sub-ring sheaf and p a point where the invariant $\nu(p, \theta, \mathcal{R})$ is infinite. Then, there exists a θ -admissible collection of local blowings-up $\tau_i : (M_i, \theta_i, \mathcal{R}_i, E_i) \rightarrow (M, \theta, \mathcal{R}, E)$ such that, for every point q_i in the pre-image of p , the invariant $\nu(q_i, \theta_i, \mathcal{R}_i)$ is finite.*

Proof. Let $\{X_1, \dots, X_d\}$ be a monomial system of generators of θ and (\mathbf{u}, \mathbf{w}) a monomial coordinate system at p . We prove the Lemma by strong induction on the number of singular vector-fields in $\{X_1, \dots, X_d\}$.

Base Step: Suppose that all vector-fields X_i are regular, i.e. that there are zero singular vector-fields on $\{X_1, \dots, X_d\}$. By the definition of monomial coordinate system, this implies that $(\mathbf{u}, \mathbf{w}) = (u_1, \dots, u_{m-d}, w_{m-d+1}, \dots, w_m)$ and we can assume that $X_j = \partial_{w_k}$ with $k = m-d+j$. So, let us consider the Taylor expansion of T_i over p :

$$T_i = \sum_{\lambda} \mathbf{w}^{\lambda} T_{i,\lambda}(\mathbf{u})$$

where $T_{i,0}$ is equivalent to zero (otherwise, it would be a first integral of θ). Now, consider the ideal \mathcal{I} generated by the functions:

$$\{T_{i,\lambda}(\mathbf{u}); \forall \lambda \text{ and } i\}$$

Since I is θ -invariant, by Theorem 2.11 there exists a sequence of θ -invariant blowings-up:

$$\tau : (\tilde{U}, \tilde{\theta}, \tilde{\mathcal{R}}, \tilde{E}) \rightarrow (U, \theta, \mathcal{R}, E)$$

that principalize I , where U is an open neighborhood of p where I is well-defined. Since the blowings-up are all θ invariant, for each point q in the pre-image of

p there exists a coordinate system (\mathbf{x}, \mathbf{w}) such that $\tau^*(X_i) = \partial_{w_k}$ and I^* is generated by a monomial \mathbf{x}^β . In particular, let (i_0, λ_0) be an index such that $T_{i_0, \lambda_0}^* = \mathbf{x}^\beta$. Thus:

$$\begin{aligned} T_{i_0}^* &= \sum_{\lambda} \mathbf{w}^\lambda T_{i_0, \lambda}(\mathbf{u})^* \\ &= \mathbf{x}^\beta \left[\mathbf{w}^{\lambda_0} U + \sum_{\lambda \neq \lambda_0} \mathbf{w}^\lambda \tilde{T}_{i_0, \lambda}(\mathbf{x}) \right] \end{aligned}$$

where U is a unit. Notice also that $\lambda_0 \neq 0$ (because $T_{i,0} \equiv 0$ for all i). So, it is clear that the invariant $\nu(q, \tilde{\theta}, \tilde{\mathcal{R}})$ is finite and smaller or equal to $\|\lambda_0\|$.

Induction Step: Suppose, by strong induction, that the Lemma is true if there are l_0 vector-fields in $\{X_1, \dots, X_d\}$ which are singular with $l_0 < l$. We assume that there are l vector-fields over $\{X_1, \dots, X_d\}$ that are singular. So, we can rename this set as $\{Y_1, \dots, Y_l, Z_{l+1}, \dots, Z_d\}$, where the vector-fields Y_i are all singular and Z_i are regular vector-fields. By the definition of monomial coordinate system, we have $(\mathbf{u}, \mathbf{w}) = (u_1, \dots, u_{m-d+l}, w_{m-d+l+1}, \dots, w_m)$ and, apart from re-indexing, we can assume that:

- $Y_j = \sum \alpha_{i,j} u_k \partial_{u_k}$ for coefficients $\alpha_{i,j} \in \mathbb{Q}$;
- $Z_j = \partial_{w_l}$ with $l = m - d + j$.

So, let us consider the Taylor expansion of T_i over p :

$$T_i = \sum_{\lambda} \mathbf{w}^\lambda T_{i, \lambda}(\mathbf{u})$$

Now, notice that, given any monomial \mathbf{u}^γ :

$$Y_j(\mathbf{u}^\gamma) = K_{j, \gamma} \mathbf{u}^\gamma$$

where $K_{j, \gamma}$ is a constant in \mathbb{Q} . Let K_γ denote the vector $(K_{1, \gamma}, \dots, K_{l, \gamma})$. In this case, we have a notion of eigenvector associated to the vector-fields Y_j :

$$T_{i, \lambda}(\mathbf{u}) = \sum_K T_{i, \lambda, K}(\mathbf{u})$$

where all monomials \mathbf{u}^γ in the expansion of $T_{i, \lambda, K}$ are such that $K_\gamma = K$. So, we can write:

$$T_i = \sum_{\lambda} \mathbf{w}^\lambda \sum_K T_{i, \lambda, K}(\mathbf{u})$$

Now, let I be the ideal generated by the functions:

$$\{T_{i, \lambda, K}; \forall i, \lambda \text{ and } K\}$$

Since I is clearly θ -invariant, by Theorem 2.11 there exists a sequence of θ -invariant blowings-up:

$$\tau : (\tilde{U}, \tilde{\theta}, \tilde{\mathcal{R}}, \tilde{E}) \rightarrow (U, \theta, \mathcal{R}, E)$$

that principalize I , where U is an open neighborhood of p where I is well-defined. Since the blowings-up are all θ -invariant, for each point q in the pre-image of p there exists a coordinate system (\mathbf{x}, \mathbf{w}) such that $\tau^*(Z_j) = \partial_{w_k}$ and I^* is generated by a monomial \mathbf{x}^β . In particular, the number of generators of $\tilde{\theta}.\mathcal{O}_q$ which are singular must be smaller or equal than l .

If the number of singular generators of q is strictly smaller than l , we can apply the strong induction hypothesis to obtain a θ -admissible collection of local blowings-up

$$\tau_q : (\tilde{U}_q, \tilde{\theta}_q, \tilde{\mathcal{R}}_q, \tilde{E}_q) \rightarrow (\tilde{U}, \tilde{\theta}, \tilde{\mathcal{R}}, \tilde{E})$$

where the invariant decreases to a finite value in the pre-image of a neighborhood of q .

So, let us assume that there exists l singular vector-fields in $\tilde{\theta}.\mathcal{O}_q$. In particular, it is clear that these vector-fields should be generated by Y_j^* (since Z_j^* are regular). Moreover, there exists an index (i_0, λ_0, K_0) such that $T_{i_0, \lambda_0, K_0}^*$ is a generator of I^* , i.e $T_{i_0, \lambda_0, K_0}^* = \mathbf{x}^\beta W$, where W is a unit. Then:

$$\begin{aligned} T_{i_0}^* &= \sum_{\lambda} \mathbf{w}^\lambda \sum_K T_{i_0, \lambda, K}(\mathbf{u})^* \\ &= \mathbf{x}^\beta \left[\mathbf{w}^{\lambda_0} \left(W + \sum_{K \neq K_0} \tilde{T}_{i_0, \lambda_0, K} \right) + \sum_{\lambda \neq \lambda_0} \mathbf{w}^\lambda \tilde{T}_{i_0, \lambda}(\mathbf{x}) \right] \end{aligned}$$

We claim that all functions $\tilde{T}_{i_0, \lambda_0, K}$ with $K \neq K_0$ are *not* unities, which implies that

$$W + \sum_{K \neq K_0} \tilde{T}_{i_0, \lambda_0, K}$$

is a unit and the invariant $\nu(q, \tilde{\theta}, \tilde{\mathcal{R}})$ is smaller or equal than $\|\lambda_0\|$. Indeed, let us assume by absurd that $\tilde{T}_{i_0, \lambda_0, K}$ is a unit for some $K \neq K_0$. In one hand, this implies that:

$$T_{i_0, \lambda_0, K}^* = \mathbf{x}^\beta V$$

where V is a unit. By another hand, since $K \neq K_0$, there exists j_0 such that the j_0 entry of K and K_0 are different. Furthermore, for any function H :

$$Y_j(H) = K_j H \implies Y_j^*(H^*) = K_j H^*$$

And, in particular:

$$\begin{aligned} Y_{j_0}^*(T_{i_0, \lambda_0, K_0}^*) &= Y_{j_0}^*(\mathbf{x}^\beta W) = \langle K_0, e_{j_0} \rangle \mathbf{x}^\beta W \text{ and} \\ Y_{j_0}^*(T_{i_0, \lambda_0, K}^*) &= Y_{j_0}^*(\mathbf{x}^\beta V) = \langle K, e_{j_0} \rangle \mathbf{x}^\beta V \end{aligned}$$

which implies that:

$$\begin{aligned} Y_{j_0}^*(\mathbf{x}^\beta) &= \mathbf{x}^\beta \left(\langle K_0, e_{j_0} \rangle + \frac{Y_{j_0}^*(W)}{W} \right) \text{ and} \\ Y_{j_0}^*(\mathbf{x}^\beta) &= \mathbf{x}^\beta \left(\langle K, e_{j_0} \rangle + \frac{Y_{j_0}^*(V)}{V} \right) \end{aligned}$$

But, since $Y_{j_0}^*$ is singular:

$$\langle K_0, e_{j_0} \rangle + \frac{Y_{j_0}^*(W)}{W} \neq \langle K, e_{j_0} \rangle + \frac{Y_{j_0}^*(V)}{V}$$

which is clearly a contradiction. Thus, the invariant is finite in an open neighborhood of q , which concludes the Lemma. \square

Lemma 4.2 (Basic normal form). *Let p be a point of M where the invariant $\nu = \nu(p, \theta, \mathcal{R})$ is finite and bigger than one, i.e $1 < \nu < \infty$. Then, there exists a system of generators (f_1, \dots, f_n) of \mathcal{R} and a monomial coordinate system $(\mathbf{u}, v, \mathbf{w})$ of p such that the functions T_i are given by:*

$$\begin{aligned} T_1 &= v^\nu U + \sum_{j=0}^{\nu-2} a_{1,j}(\mathbf{u}, \mathbf{w}) v^j \text{ where } U \text{ is an unity, and} \\ T_i &= v^\nu \bar{T}_i + \sum_{j=0}^{\nu-1} a_{i,j}(\mathbf{u}, \mathbf{w}) v^j \end{aligned} \quad (4.1)$$

and the vector-field ∂_v belongs to $\theta \cdot \mathcal{O}_p$. This Normal Form is called a Basic Normal Form.

Proof. Since the invariant is finite, it is clear that there exists a coordinate system $(\mathbf{u}, v, \mathbf{w})$ of p such that the vector-field ∂_v belongs to $\theta \cdot \mathcal{O}_p$ and for any set of generators (f_1, \dots, f_n) of \mathcal{R} , apart from re-indexing, the function $\partial_v^\nu T_1$ is a unit. Furthermore, by the implicit function Theorem, there is a change of coordinates $(\tilde{\mathbf{u}}, \tilde{v}, \tilde{\mathbf{w}}) = (\mathbf{u}, V(\mathbf{u}, v, \mathbf{w}), \mathbf{w})$ such that $\partial_{\tilde{v}}^{\nu-1} T_1(\tilde{\mathbf{u}}, 0, \tilde{\mathbf{w}}) \equiv 0$. Thus:

$$\begin{aligned} T_1 &= \tilde{v}^\nu U + \sum_{j=0}^{\nu-2} \tilde{v}^j a_{1,j}(\tilde{\mathbf{u}}, \tilde{\mathbf{w}}) \text{ where } U \text{ is an unity, and} \\ T_i &= \tilde{v}^\nu \bar{T}_i + \sum_{j=0}^{\nu-1} \tilde{v}^j a_{i,j}(\tilde{\mathbf{u}}, \tilde{\mathbf{w}}) \end{aligned}$$

Finally, since $\tilde{\mathbf{u}} = \mathbf{u}$ and $\tilde{\mathbf{w}} = \mathbf{w}$, we have that $\partial_v = U \partial_{\tilde{v}}$ for some unit U . This clearly implies that $\partial_{\tilde{v}}$ is contained in $\theta \cdot \mathcal{O}_p$, which proves the Lemma. \square

4.2 Preparation

Definition 4.3 (Prepared normal form). We say that $(M, \theta, \mathcal{R}, E)$ satisfies the Prepared Normal Form at a point p , if the invariant $\nu(p, \theta, \mathcal{R})$ is finite, and there exists a system of generators (f_1, \dots, f_n) of \mathcal{R} and monomial coordinate systems $(\mathbf{u}, v, \mathbf{w})$ of p such that the functions T_i are given by:

$$\begin{aligned} T_1 &= v^\nu U + \sum_{j=1}^{\nu-2} v^j \mathbf{u}^{\mathbf{r}_1, j} b_{1,j}(\mathbf{u}, \mathbf{w}) + b_{1,0}(\mathbf{u}, \mathbf{w}) \\ T_i &= v^\nu \bar{T}_i + \sum_{j=1}^{\nu-1} v^j \mathbf{u}^{\mathbf{r}_i, j} b_{i,j}(\mathbf{u}, \mathbf{w}) + b_{i,0}(\mathbf{u}, \mathbf{w}) \end{aligned} \quad (4.2)$$

where:

- U is an unity and \bar{T}_i are general functions;
- For $0 < j < d$ the functions $b_{i,j}$ are either units or zero. Whenever they are units, the respective exponent $\mathbf{r}_{i,j}$ is non-zero;
- Either $b_{i,0} = 0$ for all i , or there exists an index i_0 such that:

$$b_{i_0,0}(\mathbf{u}, \mathbf{w}) = \mathbf{u}^\beta w_1^\epsilon$$

where the monomial \mathbf{u}^β divides $b_{i,0}$ for all $i = 1, \dots, n$ and $\epsilon \in \{0, 1\}$.

Proposition 4.4 (Preparation). *Let p be a point of M where the invariant $\nu = \nu(p, \theta, \mathcal{R})$ is finite and bigger than one, i.e. $1 < \nu < \infty$. Furthermore, suppose that Theorem 3.7 is valid for any monomial foliated sub-ring sheaves $(N, \omega, \mathcal{S}, F)$ with $\dim N < \dim M$. Then, there exists a θ -admissible local sequence of local blowings-up $\tau_i : (M_i, \theta_i, \mathcal{R}_i, E_i) \rightarrow (M, \theta, \mathcal{R}, E)$ such that the foliated ideal sheaf $(M_i, \theta_i, \mathcal{R}_i, E_i)$ satisfies the Prepared Normal Form at every point q_i in the pre-image of p . Furthermore, $\nu(q_i, \theta_i, \mathcal{R}_i) \leq \nu(p, \theta, \mathcal{R})$.*

Remark 4.5 (Triviality of the hypothesis). Notice that, if $\dim M = 1$, the inductive hypothesis (Theorem 3.7 is valid for any monomial foliated sub-ring sheaves $(N, \omega, \mathcal{S}, F)$ with $\dim N < \dim M = 1$) is trivially true.

Proof. By Lemma 4.2, the analytic d -foliated sub-ring sheaf $(M, \theta, \mathcal{R}, E)$ satisfies the Basic Normal Form at p , i.e. there exists a system of generators (f_1, \dots, f_n) of \mathcal{R} and a monomial coordinate system $(\mathbf{u}, v, \mathbf{w})$ of p such that the functions T_i are given by (4.1) and the vector-field ∂_v belongs to $\theta \cdot \mathcal{O}_p$.

The main idea of the proof is to modify the coefficients $a_{i,j}$ without changing the v -coordinate. This is obtained through two steps, where all blowings-up are not only θ -invariant, but also ∂_v -invariant.

First Step: Let us perform a θ -admissible collection of local blowings-up to get all necessary conditions over the coefficients $a_{i,j}$ with $j > 0$. Indeed, let $\pi : M_0 \rightarrow N$ be the projection map given by $\pi(\mathbf{u}, v, \mathbf{w}) = (\mathbf{u}, \mathbf{w})$, where M_0 is a small enough neighborhood of p , and let \mathcal{J} be the principal ideal sheaf generated by the product of all non-zero $a_{i,j}$ with $j > 0$. Then, it is clear that there exist a $d - 1$ foliated ideal sheaf $(N, \omega, \mathcal{J}, F)$ such that:

- The singular distribution θ is generated by the set $\{\partial_v, \pi^* \omega\}$;
- The inverse image of F is equal to $E \cap M_0$.

Now, by Theorem 2.12 there exists a ω -admissible collection of local blowings-up

$$\sigma_i : (N_i, \omega_i, \mathcal{J}_i, F_i) \rightarrow (N, \omega, \mathcal{J}, F)$$

such that the ideal sheaf \mathcal{J}_i is monomial i.e. $\sigma_i^* \mathcal{J}$ is a principal ideal sheaf with support contained in F_i .

It is clear that we can extend σ_i to blowings-up at M_0 by taking the product of the centers of τ_i by the v -coordinate:

$$\tau_{i(1)} : (M_{i(1)}, \theta_{i(1)}, \mathcal{R}_{i(1)}, E_{i(1)}) \rightarrow (M_0, \theta_0, \mathcal{R}_0, E_0)$$

where all centers have SNC with the exceptional divisor and are invariant by the v -coordinate i.e. all centers are ∂_v -invariant. Moreover, since all centers of σ_i are ω -admissible, we conclude that all centers of $\tau_i(1)$ are θ -admissible.

Now, consider a point q_i in the pre-image of p by $\tau_i(1)$ and let $(\mathbf{u}(1), v(1), \mathbf{w}(1))$ be a coordinate system at q_i such that $\tau_i(1)^*v = v(1)$. Since the pull-back $(\tau_i(1) \circ \pi)^*\mathcal{J}$ is a principal ideal sheaf, we conclude that:

$$T_1 = v(1)^\nu U + \sum_{j=1}^{\nu-2} v(1)^j \mathbf{u}(1)^{r_{1,j}(1)} b_{1,j} + b_{1,0} \text{ where } U \text{ is an unity, and}$$

$$T_i = v(1)^\nu \bar{T}_i + \sum_{j=1}^{\nu-1} v(1)^j \mathbf{u}(1)^{r_{i,j}(1)} b_{i,j} + b_{i,0}$$

where:

- i) The functions $b_{i,j}$ are either zero or units for $j > 0$;
- ii) The monomials $\mathbf{u}(1)^{r_{i,j}(1)}$ have support in the exceptional divisor $E_i(1)$

Notice also that $\partial_{v(1)}$ clearly belongs to $\theta_i \cdot \mathcal{O}_{q_i}$ and, in particular, that $\nu(q_i, \theta_i, \mathcal{R}_i) \leq \nu(p, \theta, \mathcal{R})$.

Second Step: We now perform a θ -admissible collection of local blowings-up to get all necessary conditions over the coefficients $b_{i,0}$. Indeed, at each point q_i in the pre-image of p , apart from taking smaller varieties $M_i(1)$, there exists a projection map $\pi : M_i(1) \rightarrow N_i(1)$ given by $\pi(\mathbf{u}(1), v(1), \mathbf{w}(1)) = (\mathbf{u}(1), \mathbf{w}(1))$. Then, it is clear that there exist a $d-1$ foliated ideal sub-ring sheaf $(N_i(1), \omega_i(1), \mathcal{S}_i(1), F_i(1))$ such that:

- The singular distribution $\theta_i(1)$ is generated by the set $\{\partial_{v(1)}, \pi^*\omega_i(1)\}$;
- The inverse image of $F_i(1)$ is equal to $E_i(1)$;
- The sub-ring $\mathcal{S}_i(1)$ is generated by the restriction of all functions in $\mathcal{R}_i(1)$ to $\{v = 0\}$, i.e $\mathcal{S}_i(i)$ is generated by

$$f_k|_{\{v(1)=0\}} = g_k + \mathbf{u}^\delta b_{k,0}$$

where we recall that g_i are first integrals of $\theta_i(1)$ and, consequently, of $\omega_i(1)$ (see equation 3.1).

Notice that if all of the functions $b_{i,0} = 0$ we are done. Otherwise, the foliated sub-ring sheaf $(N_i(1), \omega_i(1), \mathcal{S}_i(1), F_i(1))$ is not trivial and, since $\dim N_i(1) < \dim M_i(1)$, we can apply Theorem 3.7 to $(N_i(1), \omega_i(1), \mathcal{S}_i(1), F_i(1))$ in order to obtain a $\omega(1)$ -admissible collection of local blowings-up

$$\sigma_{i,j}(2) : (N_{i,j}(2), \omega_{i,j}(2), \mathcal{S}_{i,j}(2), F_{i,j}(2)) \rightarrow (N_i(1), \omega_i(1), \mathcal{S}_i(1), F_i(1))$$

such that, the invariant ν calculated for $(N_{i,j}(2), \omega_{i,j}(2), \mathcal{S}_{i,j}(2), F_{i,j}(2))$ is zero or one at every point. Furthermore, by Lemma 3.6, at each point in the pre-image of q_i , there exists a coordinate $(\mathbf{u}(2), \mathbf{w}(2))$ and an index k_0 such that:

$$\begin{aligned} \sigma_{i,j}(2)^* [g_{k_0} + \mathbf{u}^\delta b_{k_0,0}] &= h_{k_0} + \mathbf{u}^\beta w_1^\epsilon \\ \sigma_{i,j}(2)^* [g_k + \mathbf{u}^\delta b_{k,0}] &= h_k + \mathbf{u}^\beta \tilde{b}_{k,0} \end{aligned} \tag{4.3}$$

where h_k is a first integral of $\omega_{i,j}(2)$ for all k , $\mathbf{u}^\beta w_1^\epsilon$ is not a first integral of $\omega_{i,j}(2)$ and $\epsilon \in \{0, 1\}$.

It is clear that we can extend $\sigma_{i,j}(2)$ to blowings-up at $M_i(2)$ by taking the product of the centers of $\tau_{i,j}(2)$ by the v -coordinate:

$$\tau_{i,j}(2) : (M_{i,j}(2), \theta_{i,j}(2), \mathcal{R}_{i,j}(2), E_{i,j}(2)) \rightarrow (M_i(1), \theta_i(1), \mathcal{R}_i(1), E_i(1))$$

where all centers have SNC with the exceptional divisor and are invariant by the v -coordinate i.e. all centers are ∂_v -invariant. Moreover, since all centers of $\sigma_{i,j}(2)$ are ω -admissible, we conclude that all centers of $\tau_{i,j}(2)$ are θ -admissible.

Now, consider a point $q_{i,j}$ in the pre-image of q_i and let $(\mathbf{u}(2), v(2), \mathbf{w}(2))$ be a monomial coordinate system of $q_{i,j}$ such that $\tau_{i,j}(2)^* v(1) = v(2)$. So, by equation (4.3)

$$\begin{aligned} \tau_{i,j}(2)^* [g_{k_0} + \mathbf{u}^\delta b_{k_0,0}] &= h_{k_0} + \mathbf{u}^\beta w_1^\epsilon \\ \tau_{i,j}(2)^* [g_k + \mathbf{u}^\delta b_{k,0}] &= h_k + \mathbf{u}^\beta \tilde{b}_{k,0} \end{aligned}$$

where h_k is a first integral of $\theta_{i,j}(2)$ for all k , $\mathbf{u}^\beta w_1^\epsilon$ is not a first integral of $\theta_{i,j}(2)$ and $\epsilon \in \{0, 1\}$. Furthermore, since all blowings-up have SNC with the exceptional divisor, we conclude that:

$$\begin{aligned} T_1 &= v(2)^\nu U + \sum_{j=1}^{\nu-2} v(2)^j \mathbf{u}(2)^{r_{1,j}(2)} c_{1,j} + c_{1,0} \text{ where } U \text{ is an unity, and} \\ T_i &= v(2)^\nu \bar{T}_i + \sum_{j=1}^{\nu-1} v(2)^j \mathbf{u}(2)^{r_{i,j}(2)} c_{i,j} + c_{i,0} \end{aligned}$$

where:

- i) The functions $c_{i,j}$ are either zero or units for $j > 0$ (this follows from [i] of the First Step);
- ii) The monomials $\mathbf{u}(2)^{r_{i,j}(2)}$ have support in the exceptional divisor $E_{i,j}(2)$ (this follows from [ii] of the Second Step);
- iii) We have that

$$\begin{aligned} c_{k_0} &= \mathbf{u}^\gamma w_1^\epsilon \\ c_k &= \mathbf{u}^\gamma \tilde{b}_{k,0} \end{aligned}$$

where $\epsilon \in \{0, 1\}$ and γ is equal to the multi-index β minus the multi-index that corresponds to the pull-back of \mathbf{u}^δ .

To finish, notice that $\partial_{v(2)}$ clearly belongs to $\theta_{i,j}(2) \cdot \mathcal{O}_{q_i}$ and, in particular, that $\nu(q_{i,j}, \theta_{i,j}, \mathcal{R}_{i,j}) \leq \nu(p, \theta, \mathcal{R})$. \square

4.3 Combinatorial Blowings-up

Definition 4.6 (Sequence of combinatorial blowings-up). Given a divisor E in M , we say that $\tau : \widetilde{M} \rightarrow M$ is a sequence of combinatorial blowings-up (with respect to E) if τ is a composition of blowings-up with centers that are strata of the divisor E and its total transforms.

Consider a monomial foliated manifold (M, θ, E) and suppose that $(\mathbf{u}, v, \mathbf{w})$ is a globally defined monomial coordinate system centered at a point p , where the vector-field ∂_v belongs to θ . We remark that, by Lemma 2.5 there exists a collection of monomials $\mathbf{u}^B = (\mathbf{u}^{\beta_1}, \dots, \mathbf{u}^{\beta_{m-d}})$ such that a vector field X on $\text{Der}_M(-\log E)$ belongs to θ if, and only if, $X(\mathbf{u}^{\beta_i}) \equiv 0$ for all i .

We now consider a sequence of combinatorial blowings-up $\tau : (\widetilde{M}, \widetilde{\theta}, \widetilde{E}) \rightarrow (M, \theta, E)$ with respect to the declared exceptional divisor $F = \{u_1 \cdots u_k \cdot v = 0\}$. Notice that such a sequence is θ -admissible and, by Theorem 2.7, $\widetilde{\theta}$ is monomial. Moreover, we can cover \widetilde{M} by affine charts with a coordinate system (\mathbf{x}, \mathbf{w}) satisfying:

$$\begin{aligned} u_j &= x_1^{a_{j,1}} \cdots x_{l+2}^{a_{j,l+1}} \\ v &= x_1^{\alpha_1} \cdots x_{l+1}^{\alpha_{l+1}} \\ w_i &= w_i \end{aligned} \tag{4.4}$$

that we denote, to simplify notation, by:

$$(\mathbf{u}, v, \mathbf{w}) = (\mathbf{x}^{\mathcal{A}}, \mathbf{w}) = (\mathbf{x}^{\mathbf{A}}, \mathbf{x}^{\boldsymbol{\alpha}}, \mathbf{w})$$

where \mathcal{A} is a $(k+1)$ -square matrix $\begin{bmatrix} \mathbf{A} \\ \boldsymbol{\alpha} \end{bmatrix}$ given by:

$$\mathbf{A} = \begin{bmatrix} a_{1,1} & \cdots & a_{1,k+1} \\ \vdots & \ddots & \vdots \\ a_{k,1} & \cdots & a_{k,k+1} \end{bmatrix} \text{ and } \boldsymbol{\alpha} = [\alpha_1 \quad \cdots \quad \alpha_{l+1}]$$

Notice that (\mathbf{x}, \mathbf{w}) is clearly a monomial coordinate system since (by Lemma 2.1)

$$\tau^* \mathbf{u}^B = \mathbf{x}^{BA}$$

which is a system of first integrals of $\widetilde{\theta}$. Now, let q be another point in this affine chart contained in the pre-image of p (recall that p is the origin of the original coordinate system). Then, apart from re-indexing,

$$(x_1, \dots, x_t, y_{t+1} + \gamma_{t+1}, \dots, y_{l+1} + \gamma_{l+1}, \mathbf{w}),$$

is a coordinate system centered at q , and $\gamma_j \neq 0$ for all the γ_j . We can also assume that $t \neq 0$, otherwise q would be outside the exceptional divisor (and, thus, outside the inverse image of p). In this case, we have a decomposition of the matrix \mathcal{A}

$$\mathcal{A} = \begin{bmatrix} \mathbf{A}_1 & \mathbf{A}_2 \\ \boldsymbol{\alpha}_1 & \boldsymbol{\alpha}_2 \end{bmatrix}$$

where \mathbf{A}_1 is a $k \times t$ matrix, \mathbf{A}_2 is a $k \times (k+1-t)$ matrix, $\boldsymbol{\alpha}_1$ is a $1 \times t$ matrix and $\boldsymbol{\alpha}_2$ is a $1 \times (k-t+1)$ matrix. We remark that, since q is a point on the exceptional divisor \widetilde{E} , there exists at least one u_i such that $\tau^* u_i(q) = 0$, which clearly implies that \mathbf{A}_1 has to be a non-zero matrix.

Lemma 4.7 (Claim 1). *Assume \mathbf{A}_1 has maximal rank. Then, there exists a monomial coordinate system $(\mathbf{x}, \mathbf{y}, z, \mathbf{w}) = (x_1, \dots, x_t, y_{t+1}, \dots, y_l, z, \mathbf{w})$ centered at q such that*

$$\begin{aligned} \mathbf{u} &= \mathbf{x}^{\mathbf{A}_1} (\mathbf{y} - \widetilde{\gamma})^{\boldsymbol{\Lambda}} \\ v &= \mathbf{x}^{\boldsymbol{\alpha}_1} (z - \widetilde{\gamma}) \\ \mathbf{w} &= \mathbf{w} \end{aligned} \tag{4.5}$$

where $\tilde{\gamma}_j \neq 0$ for all j and the matrix $\mathbf{\Lambda} = (\lambda_{i,j})$ of exponents has maximal rank, with $\lambda_{i,j} \in \mathbb{Q}$ (in particular, ∂_z is contained in $\tilde{\theta}.\mathcal{O}_q$). Moreover, if $\mathbf{u}^{\mathbf{\xi}}$ is **not** a first integral of $\theta.\mathcal{O}_p$, then its total transform $\mathbf{u}^{\mathbf{\xi}} = \mathbf{x}^{\tilde{\mathbf{\xi}}}U$, where U is a unit, satisfies one of the following:

- Either the monomial $\mathbf{x}^{\tilde{\mathbf{\xi}}}$ is **not** a first integral of $\tilde{\theta}$;
- Or, there exists a regular vector-field $\partial_{y_i} \in \tilde{\theta}.\mathcal{O}_q$ such that $\partial_{y_i}U$ is a unit.

Lemma 4.8 (Claim 2). Assume that \mathbf{A}_1 does not have maximal rank. Then, there exists a monomial coordinate system $(\mathbf{x}, \mathbf{y}, \mathbf{w}) = (x_1, \dots, x_t, y_{t+1}, \dots, y_{l+1}, \mathbf{w})$ centered at q such that

$$\begin{aligned} \mathbf{u} &= \mathbf{x}^{\mathbf{A}_1}(\mathbf{y} - \boldsymbol{\lambda})^{\mathbf{\Lambda}} \\ v &= \mathbf{x}^{\boldsymbol{\alpha}_1} \\ \mathbf{w} &= \mathbf{w} \end{aligned} \tag{4.6}$$

where $\tilde{\gamma}_j \neq 0$ for all j and the matrix $\mathbf{\Lambda} = (\lambda_{i,j})$ of exponents has maximal rank and $\boldsymbol{\alpha}_1$ doesn't belong to the span of the rows of \mathbf{A}_1 . Moreover, if $\mathbf{u}^{\mathbf{\xi}}$ is **not** a first integral of $\theta.\mathcal{O}_p$, then its total transform $\mathbf{u}^{\mathbf{\xi}} = \mathbf{x}^{\tilde{\mathbf{\xi}}}U$, where U is a unit, satisfies one of the following:

- Either the monomial $\mathbf{x}^{\tilde{\mathbf{\xi}}}$ is **not** a first integral of $\tilde{\theta}$;
- Or, there exists a regular vector-field $\partial_{y_i} \in \tilde{\theta}.\mathcal{O}_q$ such that $\partial_{y_i}U$ is a unit.

Proof of Lemma 4.7. By hypothesis, apart from re-indexing of the u_j 's, we can write

$$\mathbf{A}_1 = \begin{bmatrix} \mathbf{A}'_1 \\ \mathbf{A}''_1 \end{bmatrix}, \quad \mathbf{A}_2 = \begin{bmatrix} \mathbf{A}'_2 \\ \mathbf{A}''_2 \end{bmatrix},$$

where $\det(\mathbf{A}'_1) \neq 0$, and \mathbf{A}'_1 and \mathbf{A}'_2 have the same height. So, we can write equations (4.4) in the compact form

$$\begin{aligned} \mathbf{u}' &= \mathbf{x}^{\mathbf{A}'_1}(\mathbf{y} - \boldsymbol{\gamma})^{\mathbf{A}'_2} \\ \mathbf{u}'' &= \mathbf{x}^{\mathbf{A}''_1}(\mathbf{y} - \boldsymbol{\gamma})^{\mathbf{A}''_2} \\ v &= \mathbf{x}^{\boldsymbol{\alpha}_1}(\mathbf{y} - \boldsymbol{\gamma})^{\boldsymbol{\alpha}_2} \end{aligned}$$

First change of coordinates:

$$\begin{aligned} \mathbf{x}(1) &= \mathbf{x} \cdot (\mathbf{y} - \boldsymbol{\gamma})^{(\mathbf{A}'_1)^{-1} \mathbf{A}'_2} \\ \mathbf{y}(1) &= \mathbf{y} \end{aligned}$$

After this change of coordinates we get (using Lemma 2.1)

$$\begin{aligned} \mathbf{u}' &= \mathbf{x}(1)^{\mathbf{A}'_1} \\ \mathbf{u}'' &= \mathbf{x}(1)^{\mathbf{A}''_1}(\mathbf{y}(1) - \boldsymbol{\gamma})^{\mathbf{\Lambda}(1)} \\ v &= \mathbf{x}(1)^{\boldsymbol{\alpha}_1}(\mathbf{y}(1) - \boldsymbol{\gamma})^{\boldsymbol{\lambda}(1)} \end{aligned} \tag{4.7}$$

and the square matrix $\mathcal{L} := \begin{bmatrix} \Lambda^{(1)} \\ \lambda^{(1)} \end{bmatrix} := \begin{bmatrix} \mathbf{A}_2'' - \mathbf{A}_1''(\mathbf{A}_1')^{-1}\mathbf{A}_2' \\ \alpha_2 - \alpha_1(\mathbf{A}_1')^{-1}\mathbf{A}_2' \end{bmatrix}$ has determinant different from zero because the full matrix of exponents in (4.7) is obtained from \mathcal{A} by a sequence of column elementary transformations. Notice also that the entries of $(\mathbf{A}_1')^{-1}\mathbf{A}_2'$ are rational numbers, not necessarily integers.

Second change of coordinates: After reindexing the $y_{i(1)} - \gamma_i$ we can assume that the elements of the diagonal of \mathcal{L} are different from zero. Thus, we consider

$$\begin{aligned} \mathbf{y}^{(2)} - \gamma^{(2)} &= (\mathbf{y}^{(1)} - \gamma)^{\Lambda^{(1)}} \\ (z^{(2)} - \gamma_{l+1}^{(2)}) &= (\mathbf{y}^{(1)} - \gamma)^{\lambda^{(1)}} \\ \mathbf{x}^{(2)} &= \mathbf{x}^{(1)} \end{aligned} \tag{4.8}$$

so to get

$$\begin{aligned} \mathbf{u}' &= \mathbf{x}^{(2)} \mathbf{A}_1' \\ \mathbf{u}'' &= \mathbf{x}^{(2)} \mathbf{A}_1'' (\mathbf{y}^{(2)} - \gamma^{(2)}) \mathbf{Id} \\ v &= \mathbf{x}^{(2)} \alpha_1 (z^{(2)} - \gamma_{l+1}^{(2)}) \end{aligned}$$

where \mathbf{Id} is the identity matrix.

Third change of coordinates: We need to guarantee that the coordinate system is monomial. Thus, let us now recall the first integrals \mathbf{u}^B . Notice that the matrix B can be written as:

$$B = [\mathbf{B}_1 \quad \mathbf{B}_2]$$

where \mathbf{B}_2 is a $k \times (k - t)$ matrix. With this notation, by Lemma 2.1:

$$\mathbf{u}^B = \mathbf{x}^{(2)} \mathbf{B} \mathbf{A}_1 (\mathbf{y}^{(2)} - \gamma^{(2)})^{\mathbf{B}_2} = \mathbf{x}^{(2)} \mathbf{C}_1 (\mathbf{y}^{(2)} - \gamma^{(2)})^{\mathbf{C}_2}$$

where $\mathbf{C}_1 = \mathbf{B} \mathbf{A}_1$ is a non-zero matrix (since B and \mathbf{A}_1 are of maximal rank) and $\mathbf{C}_2 = \mathbf{B}_2$. Now, we perform a change of coordinates similar with the one given in Lemma 2.6 in order to obtain a monomial coordinate system. To that end, consider:

$$C = \begin{bmatrix} C_1' & C_2' \\ C_1'' & C_2'' \end{bmatrix}$$

where $C_1 = \begin{bmatrix} C_1' \\ C_1'' \end{bmatrix}$ and the rank of C_1' is maximal and equal to the rank of C_1 .

So, there exists a change of coordinates $(\mathbf{x}^{(3)}, \mathbf{y}^{(3)}, z^{(3)}, \mathbf{w}^{(3)})$ where $z^{(3)} = z^{(2)}$, such that:

$$\mathbf{u}^B = \mathbf{x}^{(3)} \mathbf{D}_1 (\mathbf{y}^{(3)} - \gamma^{(3)})^{\mathbf{D}_2}$$

where:

$$D = [\mathbf{D}_1 \quad \mathbf{D}_2] = \begin{bmatrix} \mathbf{D}_1' & \mathbf{D}_2' \\ \mathbf{D}_1'' & \mathbf{D}_2'' \end{bmatrix} = \begin{bmatrix} C_1' & 0 \\ C_1'' & \Delta \end{bmatrix}$$

where Δ is a maximal rank matrix of rational numbers. This implies that the collection $(\mathbf{x}^{(3)} \mathbf{D}_1', \mathbf{x}^{(3)} \mathbf{D}_1'' (\mathbf{y}^{(3)} - \gamma^{(3)})^\Delta)$ is a collection of first integrals of $\theta \cdot \mathcal{O}_q$.

Forth change of coordinates: Let r be the rank of Δ . Then, apart from re-ordering the $\mathbf{y}^{(3)}$ coordinates, there exists a coordinate system $(\mathbf{x}^{(4)}, \mathbf{y}^{(4)}, \mathbf{v}^{(4)}, z^{(4)})$,

$\mathbf{w}(4)$) where $\mathbf{x}(4) = \mathbf{x}(3)$, $z(4) = z(3)$, $\mathbf{w}(4) = \mathbf{w}(3)$ and $\mathbf{v}(4) = (y_{t+r+1}(3), \dots, y_{l+1}(3))$ such that:

$$(\mathbf{y}(3) - \gamma(3))^\Delta = \mathbf{y}(4) - \gamma_y(4)$$

where $\mathbf{y}(4) = (y_{t+1}(4), \dots, y_r(4))$, which implies that the monomial functions

$$(\mathbf{x}(4))^{D'_1}, \mathbf{y}(4)$$

are first integrals of $\theta.\mathcal{O}_q$, which guarantees that the coordinate system is monomial. Furthermore, since $z(2) = z(4)$ we finally conclude that:

$$\begin{aligned} \mathbf{u} &= \mathbf{x}(4)^{\mathbf{A}_1} (\mathbf{y}(4) - \gamma_y(4))^{\mathbf{A}_y(4)} (\mathbf{v}(4) - \gamma_v(4))^{\mathbf{A}_v(4)} \\ v &= \mathbf{x}(4)^{\alpha_1} (\mathbf{y}(4) - \gamma_y(4))^{\lambda_y(4)} (\mathbf{v}(4) - \gamma_v(4))^{\lambda_v(4)} (z(4) - \gamma_{\gamma_{l+1}}(4)) \\ \mathbf{w} &= \mathbf{w} \end{aligned}$$

where $\mathbf{\Lambda}(4) = [\mathbf{\Lambda}_y(4), \mathbf{\Lambda}_v(4)]$ is a maximal rank matrix of rational numbers.

Fifth change of coordinates: It is now clear that we only need a change in the $z(4)$ so that

$$z(5) - \gamma_{\gamma_{l+1}}(5) = (\mathbf{y}(4) - \gamma_y(4))^{\lambda_y(4)} (\mathbf{v}(4) - \gamma_v(4))^{\lambda_v(4)} (z(4) - \gamma_{\gamma_{l+1}}(4))$$

which clearly does not change the fact that the coordinate system is monomial. This is the coordinate system of the enunciate of the Lemma.

Now, let \mathbf{u}^ξ be a monomial which is **not** a first integral of $\theta.\mathcal{O}_p$, i.e., the multi-index ξ doesn't belong to the span of the rows of \mathbf{B} . In this case, let us notice that:

$$\mathbf{u}^\xi = \mathbf{x}(5)^{\xi \mathbf{A}_1} (\mathbf{y}(5) - \gamma_y(5))^{\xi \mathbf{A}_y(5)} (\mathbf{v}(5) - \gamma_v(5))^{\xi \mathbf{A}_v(5)}$$

Now, we claim that either $\xi \mathbf{A}_1$ is a linearly independent vector with the span of the rows of \mathbf{D}'_1 or $\xi \mathbf{A}_{v(5)} \neq 0$, which concludes the Lemma. Indeed, let us suppose, by contraction that $\xi \mathbf{A}_1$ is spanned by the rows of \mathbf{D}'_1 and $\xi \mathbf{A}_{v(5)} = 0$. In this case, it is clear that the vector $\xi[\mathbf{A}_1, \mathbf{\Lambda}(5)]$ is spanned by the rows of \mathbf{D} , which clearly contradicts the fact that ξ is not in the span of the rows of \mathbf{B} . This finishes the proof. \square

Proof of Lemma 4.8. We have

$$\begin{aligned} \mathbf{u}^B &= \mathbf{x}^{B \mathbf{A}_1} (\mathbf{y} - \gamma)^{B \mathbf{A}_2} \\ \mathbf{u} &= \mathbf{x}^{\mathbf{A}_1} (\mathbf{y} - \gamma)^{\mathbf{A}_2} \\ v &= \mathbf{x}^{\alpha_1} (\mathbf{y} - \gamma)^{\alpha_2}, \end{aligned}$$

Notice that, since \mathbf{A} is of maximal rank but \mathbf{A}_1 does not have maximal rank, α_1 doesn't belong to the span of the rows of \mathbf{A}_1 . Thus, it does not belong to the span of the rows of $\mathbf{B} \mathbf{A}_1$.

First change of coordinates: There exists a coordinate system $(\mathbf{x}(1), \mathbf{y}(1), \mathbf{w}(1))$ where $\mathbf{y}(1) = \mathbf{y}$ and $\mathbf{w}(1) = \mathbf{w}$ such that

$$\begin{aligned} \mathbf{u}^B &= \mathbf{x}(1)^{C_1} (\mathbf{y}(1) - \gamma)^{C_2} \\ \mathbf{u} &= \mathbf{x}(1)^{\mathbf{A}_1} (\mathbf{y}(1) - \gamma)^{\mathbf{A}(1)} \\ v &= \mathbf{x}(1)^{\alpha_1} \end{aligned}$$

where $C = \begin{bmatrix} C_1 \\ C_2 \end{bmatrix} = \begin{bmatrix} BA_1 \\ C_2 \end{bmatrix}$ and $\Lambda_{(1)}$ are matrices of maximal rank.

Second change of coordinates: We need to guarantee that the coordinate system is monomial. To that end, consider:

$$C = \begin{bmatrix} C'_1 & C'_2 \\ C''_1 & C''_2 \end{bmatrix}$$

where $C_1 = \begin{bmatrix} C'_1 \\ C''_1 \end{bmatrix}$ and the rank of C'_1 is maximal and equal to the rank of C_1 .

Since α_1 does not belong to the span of the rows of C'_1 , there exists a change of coordinates $(\mathbf{x}(2), \mathbf{y}(2), \mathbf{w}(2))$ where $v = \mathbf{x}(2)^{\alpha_1}$, such that:

$$\begin{aligned} \mathbf{u}^B &= \mathbf{x}(2)^{D_1} (\mathbf{y}(2) - \gamma(2))^{D_2} \\ \mathbf{u} &= \mathbf{x}(2)^{A_1} (\mathbf{y}(2) - \gamma(2))^{\Lambda(2)} \\ v &= \mathbf{x}(2)^{\alpha_1} \end{aligned}$$

where $\Lambda(2)$ is a maximal rank matrix of rational numbers and

$$D = [D_1 \quad D_2] = \begin{bmatrix} D'_1 & D'_2 \\ D''_1 & D''_2 \end{bmatrix} = \begin{bmatrix} C'_1 & 0 \\ C''_1 & \Delta \end{bmatrix}$$

where Δ is a maximal rank matrix of rational numbers. This implies that the collection $(\mathbf{x}(2)^{D'_1}, \mathbf{x}(2)^{D''_1} (\mathbf{y}(2) - \gamma(2))^{\Delta})$ is a collection of first integrals of $\theta.\mathcal{O}_q$. Since D'_1 has rank equal to D_1 , we conclude that:

$$(\mathbf{x}(2)^{D'_1}, (\mathbf{y}(2) - \gamma(2))^{\Delta})$$

is another collection of first integrals of $\theta.\mathcal{O}_q$.

Third change of coordinates: Since Δ is of maximal rank, there exists a coordinate system $(\mathbf{x}(3), \mathbf{y}(3), \mathbf{z}(3), \mathbf{w}(3))$ where $\mathbf{x}(3) = \mathbf{x}(2)$ and $\mathbf{w}(3) = \mathbf{w}(2)$ such that:

$$(\mathbf{y}(2) - \gamma(2))^{\Delta} = \mathbf{y}(3) - \gamma(3)$$

which finally implies that the monomial functions

$$(\mathbf{x}(3)^{D'_1}, \mathbf{y}(3))$$

are first integrals of $\theta.\mathcal{O}_q$. This implies that this coordinate system is monomial. Furthermore, since $\mathbf{x}(2)^{\alpha_1}$ is independent of the $\mathbf{y}(2)$ coordinate, we finally conclude that:

$$\begin{aligned} \mathbf{u} &= \mathbf{x}(3)^{A_1} (\mathbf{y}(3) - \gamma_1(3))^{\Lambda_1(3)} (\mathbf{z}(3) - \gamma_2(3))^{\Lambda_2(3)} \\ v &= \mathbf{x}(3)^{\alpha_1} \\ \mathbf{w} &= \mathbf{w}(3) \end{aligned}$$

where $\Lambda = [\Lambda_1 \quad \Lambda_2]$ is a maximal rank matrix of rational numbers and $\gamma(3) = (\gamma_1(3), \gamma_2(3))$ is a vector where no entry is zero. This proves that the coordinate system is monomial.

Now, let \mathbf{u}^ξ be a monomial which is **not** a first integral of $\theta.\mathcal{O}_p$, i.e., the multi-index ξ doesn't belong to the span of the rows of \mathbf{B} . In this case, let us notice that:

$$\mathbf{u}^\xi = \mathbf{x}_{(3)}^{\xi \mathbf{A}_1} (\mathbf{y}_{(3)} - \gamma_1(3))^{\xi \Lambda_1(3)} (\mathbf{z}_{(3)} - \gamma_2(3))^{\xi \Lambda_2(3)}$$

Now, we claim that either $\xi \mathbf{A}_1$ is a linearly independent vector with the span of the rows of \mathbf{D}'_1 or $\xi \Lambda_{2(3)} \neq 0$, which concludes the Lemma. Indeed, let us suppose, by contraction that $\xi \mathbf{A}_1$ is spanned by the rows of \mathbf{D}'_1 and $\xi \Lambda_{v(3)} = 0$. In this case, it is clear that the vector $\xi[\mathbf{A}_1, \Lambda(3)]$ is spanned by the rows of \mathbf{D} , which clearly contradicts the fact that ξ is not in the span of the rows of \mathbf{B} . This finishes the proof. \square

4.4 Dropping the invariant from Prepared Normal Form

Lemma 4.9. *Let $(M, \theta, \mathcal{R}, E)$ be an analytic foliated sub ring sheaf that satisfies the Prepared Normal Form at a point p where the invariant $\nu = \nu(p, \theta, \mathcal{R})$ is finite and bigger than one, i.e $1 < \nu < \infty$. Then, for a small enough neighborhood M_0 of p , there exists a sequence of θ -admissible blowings-up $\tau : (M_r, \theta_r, \mathcal{R}_r, E_r) \rightarrow (M_0, \theta_0, \mathcal{R}_0, E_0)$ such that, for all point q in the pre-image of p , the invariant $\nu(q, \theta_r, \mathcal{R}_r)$ is strictly smaller than the initial invariant $\nu(p, \theta, \mathcal{R})$.*

Proof. By hypothesis, there exists a local coordinate system $(\mathbf{u}, v, \mathbf{w})$ that satisfies the Prepared Normal Form at p with $\nu = \nu_p(\theta, \mathcal{R})$, i.e. that satisfies equations (4.2). Since θ is monomial, by Lemma 2.5, there exists $m - d$ monomials $\mathbf{u}^{\mathbf{B}} = (\mathbf{u}^{\beta_1}, \dots, \mathbf{u}^{\beta_{m-d}})$, such that

$$\theta.\mathcal{O}_p = \{X \in \text{Der}_p(-\log E); X(\mathbf{u}^{\beta_i}) \equiv 0 \text{ for all } i\}$$

Let us now consider the ideal \mathcal{J} generated by:

$$v^\nu, \text{ and } \{v^j \mathbf{u}^{\mathbf{r}_{i,j}} b_{i,j}\}_{1 \leq j < d}, \text{ and } \mathbf{u}^\beta$$

where we recall that all $b_{i,j}$ are either units or zero for $j > 0$. Notice that we include only the monomial \mathbf{u}^β and not $\mathbf{u}^\beta w_1^\epsilon$ in the ideal. Now, consider a sequence of blowings-up:

$$\tau : (M_r, \theta_r, \mathcal{R}_r, E_r) \rightarrow (M_0, \theta_0, \mathcal{R}_0, E_0)$$

that principalize \mathcal{J} , where M_0 is any fixed open neighborhood of p where \mathcal{J} is well-defined. Since \mathcal{J} is generated by monomials in the variables \mathbf{u} and v , this sequence can be chosen to be combinatorial with respect to the divisor $F := \{u_1 \cdots u_l v = 0\}$ (see Definition 4.6). Furthermore, we know that the sequence τ is θ -admissible.

Now, let q be a point of M_r in the pre-image of p . We claim that $\nu(q, \theta_r, \mathcal{R}_r) < \nu(p, \theta, \mathcal{R})$, which is enough to conclude the Lemma. Indeed, since τ is a sequence of combinatorial blowings-up in respect to the divisor F , the point q satisfies the hypothesis of either Lemma 4.7 or 4.8. Thus, we have two cases to consider:

Case 1: We assume we are in conditions of Lemma 4.7. There exists a monomial system of coordinates $(\mathbf{x}, \mathbf{y}, z, \mathbf{w}) = (x_1, \dots, x_t, y_{t+1}, \dots, y_l, z, \mathbf{w})$ centered at q such that

$$\begin{aligned} \mathbf{u} &= \mathbf{x}^{\mathbf{A}_1}(\mathbf{y} - \tilde{\gamma})^{\mathbf{A}} \\ v &= \mathbf{x}^{\alpha_1}(z - \tilde{\gamma}_{l+1}) \\ \mathbf{w} &= \mathbf{w} \end{aligned} \quad (4.9)$$

where $\tilde{\gamma}_j \neq 0$ for all j and the matrix $\mathbf{A} = (\lambda_{i,j})$ of exponents has maximal rank, with $\lambda_{i,j} \in \mathbb{Q}$. In particular, ∂_z is contained in $\tilde{\theta} \cdot \mathcal{O}_q$ (this follows from the above coordinate change). So, after blowing-up we have the following expressions:

$$\begin{aligned} \tau^* T_1 &= U \mathbf{x}^{S_\nu} (z - \tilde{\gamma}_{l+1})^\nu + \sum_{i=1}^{\nu-1} \mathbf{x}^{S_{1,j}} (z - \tilde{\gamma}_{l+1})^i c_{1,j}(\mathbf{x}, \mathbf{y}, \mathbf{w}) + \mathbf{x}^{S_0} c_{1,0}(\mathbf{x}, \mathbf{y}, \mathbf{w}) \\ \tau^* T_i &= \tilde{T}_i \mathbf{x}^{S_\nu} (z - \tilde{\gamma}_{l+1})^\nu + \sum_{j=1}^{\nu-1} \mathbf{x}^{S_{i,j}} (z - \tilde{\gamma}_{l+1})^j c_{i,j}(\mathbf{x}, \mathbf{y}, \mathbf{w}) + \mathbf{x}^{S_0} c_{i,0}(\mathbf{x}, \mathbf{y}, \mathbf{w}) \end{aligned} \quad (4.10)$$

where:

- The function U is a unit of the form $\tilde{U}(\mathbf{x}, \mathbf{y}, \mathbf{w}) + \mathbf{x}^{\alpha_1} \Omega(\mathbf{x}, \mathbf{y}, z, \mathbf{w})$, where $\tilde{U}(\mathbf{x}, \mathbf{y}, \mathbf{w})$ is a unit and $\alpha_1 \neq 0$ (because q is in the pre-image of p);
- For $j > 0$ the functions $c_{i,j}$ are either zero or units (that don't depend on z);
- The term $\mathbf{x}^{S_0} c_{i,0}$ is the pullback of $b_{i,0}$. In particular, either $c_{i,0} = 0$ for all i , or the i_0 -term $\mathbf{x}^{S_0} c_{i_0,0}$ is equal to $\mathbf{x}^{S_0} w_2^\epsilon \tilde{c}_{i_0,0}$ where $\tilde{c}_{i_0,0}$ is a unit.

We consider three cases depending on which generator of \mathcal{I} pulls back to be a generator of the pull-back of \mathcal{I}^* :

Case 1.1: [The pull back of v^ν generates \mathcal{J}^* , i.e. $S_\nu = \min\{S_\nu, S_0, S_{i,j}\}$] In this case, by equation (4.10), we have:

$$\begin{aligned} \tau^* T_1 &= \mathbf{x}^{S_\nu} \left[\left(\tilde{U} z + \tilde{U} \tilde{\gamma}_{l+1} \nu + \mathbf{x}^{\alpha_1} \Omega_2 \right) z^{\nu-1} + \right. \\ &\quad \left. + \text{terms where the exponent of } z \text{ is } < \nu - 1 \right] \end{aligned}$$

where α_1 is a non-zero matrix and $\Omega_2 = [z + \tilde{\gamma}_{l+1} \nu] \Omega$. Since $\tilde{U} z + \tilde{U} \nu \tilde{\gamma}_{l+1} + \mathbf{x}^{\alpha_1} \Omega_2$ is a unit and the vector-field ∂_z belongs to θ_r , it is clear that $\nu(q, \theta_r, \mathcal{R}_r) \leq \nu - 1$.

Case 1.2: [There is a maximum $0 < j_1 < d$ such that the pull back of $u^{r_{i_1, j_1}} v^i$ generates \mathcal{J}^* for some i_1 , i.e. $S_{i_1, j_1} = \min\{S_\nu, S_0, S_{i,j}\}$, $S_\nu > S_{i_1, j_1}$ and $S_{i,j} > S_{i_1, j_1}$ for $j > j_1$]. In this case, by equation (4.10), we have:

$$\tau^* T_{i_1} = \mathbf{x}^{S_{i_1, j_1}} \left[(z - \tilde{\gamma}_{l+1})^{j_1} c_{i_1, j_1} + \sum_{j=0}^{j_1-1} \mathbf{x}^{S_{i_1, j} - S_{i_1, j_1}} (z - \tilde{\gamma}_{l+1})^j c_{i_1, j} + \Omega(\mathbf{x}, \mathbf{y}, z, \mathbf{w}) \right]$$

where $\Omega(\mathbf{0}, \mathbf{y}, z, \mathbf{w}) \equiv 0$. Since c_{i_1, j_1} is a unit and the vector-field ∂_z belongs to θ_r , it is clear that $\nu(q, \theta_r, \mathcal{R}_r) \leq j_1 < \nu$.

Case 1.3: [The pull-back of \mathbf{u}^β is the *only* generator of \mathcal{J}^* i.e. $S_0 = \min\{S_\nu, S_0, S_{i,j}\}$ and $S_\nu > S_0$, $S_{i,j} > S_0$] In this case, we recall that there exists i_0 such that

$$\mathbf{x}^{S_0} c_{i_0,0} = \mathbf{x}^{S_0} w_2^\epsilon W$$

where W is a unit. We consider two cases depending on ϵ :

Case 1.3a, $\epsilon = 1$: Then

$$\tau^* T_{i_0} = \mathbf{x}^{S_0} [w_2 W + \Omega(\mathbf{x}, \mathbf{y}, z, \mathbf{w})]$$

where W is a unit and $\Omega(\mathbf{0}, \mathbf{y}, z, \mathbf{w}) \equiv 0$. Since the vector-field ∂_{w_2} clearly belongs to $\theta_r \mathcal{O}_q$, we conclude that $\nu(q, \theta_r, \mathcal{R}_r) \leq 1 < \nu$.

Case 1.3b, $\epsilon = 0$: In this case, notice that the monomial $\mathbf{u}^{\beta+\delta}$ is **not** a first integral of $\theta \mathcal{O}_p$. Thus, Lemma 4.7 guarantees that the total transform $\mathbf{u}^{\beta+\delta} = \mathbf{x}^{S_0+\tilde{\delta}} \tilde{W}$, where $\tilde{W} = W(\mathbf{y} - \boldsymbol{\gamma})^{\delta \Lambda}$ is a unit, satisfies one of the following:

- Either $\mathbf{x}^{S_0+\tilde{\delta}}$ is **not** a first integral of $\tilde{\theta} \mathcal{O}_q$, which implies that

$$\tau^* [\mathbf{u}^{\delta} T_{i_0}] = \mathbf{x}^{S_0+\tilde{\delta}} U$$

for some unit U . We conclude that $\nu(q, \theta_r, \mathcal{R}_r) = 0 < \nu$;

- Or, there exists a regular vector-field $\partial_{y_i} \in \tilde{\theta} \mathcal{O}_q$ such that $\partial_{y_i} \tilde{W}$ is a unit. In particular:

$$\tau^* [\mathbf{u}^{\delta} T_{i_0}] = \mathbf{x}^{S_0+\tilde{\delta}} \tilde{W}(0) + \mathbf{x}^{S_0+\tilde{\delta}} [\tilde{W} - \tilde{W}(0) + \Omega(\mathbf{x}, \mathbf{y}, z, \mathbf{w})]$$

where $\Omega(\mathbf{0}, \mathbf{y}, z, \mathbf{w}) \equiv 0$ and the monomial $\mathbf{x}^{S_0+\tilde{\delta}} \tilde{W}(0)$ is a first integral of $\tilde{\theta} \mathcal{O}_q$. We conclude that $\nu(q, \theta_r, \mathcal{R}_r) \leq 1 < \nu$.

Case 2: We assume we are in conditions of Lemma 4.8. There exists a monomial system of coordinates $(\mathbf{x}, \mathbf{y}, \mathbf{w}) = (x_1, \dots, x_t, y_{t+1}, \dots, y_{l+1}, \mathbf{w})$ centered at q such that

$$\begin{aligned} \mathbf{u} &= \mathbf{x}^{A_1} (\mathbf{y} - \boldsymbol{\lambda})^\Lambda \\ v &= \mathbf{x}^{\alpha_1} \\ \mathbf{w} &= \mathbf{w} \end{aligned} \tag{4.11}$$

where $\tilde{\gamma}_j \neq 0$ for all j and the matrix $\boldsymbol{\Lambda} = (\lambda_{i,j})$ of exponents has maximal rank and α_1 doesn't belong to the span of the rows of A_1 . So, after blowing-up we have the following expressions:

$$\begin{aligned} \tau^* T_1 &= U \mathbf{x}^{S_\nu} + \sum_{j=1}^{\nu-1} \mathbf{x}^{S_{1,j}} c_{1,j}(\mathbf{x}, \mathbf{y}, \mathbf{w}) + \mathbf{x}^{S_0} c_{1,0}(\mathbf{x}, \mathbf{y}, \mathbf{w}) \\ \tau^* T_i &= \tilde{T}_i \mathbf{x}^{S_\nu} + \sum_{j=1}^{\nu-1} \mathbf{x}^{S_{i,j}} c_{i,j}(\mathbf{x}, \mathbf{y}, \mathbf{w}) + \mathbf{x}^{S_0} c_{i,0}(\mathbf{x}, \mathbf{y}, \mathbf{w}) \end{aligned} \tag{4.12}$$

where:

- The function U is a unit and, for $j > 0$, the functions $c_{i,j}$ are either zero or units (that don't depend on z);

- The term $\mathbf{x}^{S_0} c_{i,0}$ is the pullback of $b_{i,0}$. In particular, either $c_{i,0} = 0$ for all i , or the i_0 -term $\mathbf{x}^{S_0} c_{i_0,0}$ is equal to $\mathbf{x}^{S_0} w_2^\epsilon \tilde{c}_{i_0,0}$ where $\tilde{c}_{i_0,0}$ is a unit;
- We remark that:

$$S_\nu = \nu \alpha_1$$

$$S_{i,j} = j \alpha_1 + \mathbf{r}_{i,j} \mathbf{A}_1, \text{ for } i, j = 0, \dots, \nu - 1.$$

So, for a fixed i , each S_ν and $S_{i,j}$ is a sum of an element of the span of the rows of \mathbf{A}_1 and a different multiple of the α_1 . Since α_1 is linearly independent with the rows of \mathbf{A}_1 , this means that the exponents S_ν and $S_{i,j}$ are all distinct. Therefore, for each i fixed, all of the multi-indexes $S_{i,j}$ must be different.

We consider three cases depending on which generator of \mathcal{I} pulls back to be a generator of the pull-back of \mathcal{I}^* :

Case 2.1: [The pull back of v^ν generates \mathcal{J}^* , i.e. $S_\nu = \min\{S_\nu, S_0, S_{i,j}\}$, $S_\nu < S_0$ and $S_\nu < S_{i,j}$ for all (i, j)] In this case, from equation (4.12), we have:

$$\tau^*[\mathbf{u}^\delta T_1] = \mathbf{x}^{S_\nu + \delta} [\tilde{U} + \Omega(\mathbf{x}, \mathbf{y}, z, \mathbf{w})]$$

where $\tilde{\delta} = \delta \mathbf{A}_1$, the function $\tilde{U} = U(\mathbf{y} - \gamma)^{\delta \Lambda}$ is a unit and $\Omega(\mathbf{0}, \mathbf{y}, z, \mathbf{w}) \equiv 0$. Since $\mathbf{x}^{S_\nu + \delta}$ is not a first integral of $\tilde{\theta}$ (which is clear from that fact that α_1 doesn't belong to the span of the rows of \mathbf{A}_1), we conclude that $\nu(q, \theta_r, \mathcal{R}_r) = 0 < \nu$. (Must talk about the δ).

Case 2.2: [There is a maximum $0 < j_1 < \nu$ such that the pull back of $u^{r_{i_1, j_1}} v^i$ is a generator of \mathcal{J}^* for some i_1 , i.e. $S_{i_1, j_1} = \min\{S_\nu, S_0, S_{i,j}\}$, $S_\nu > S_{i_1, j_1}$ and $S_{i_1, j} > S_{i_1, j_1}$ for all j]. In this case, from equation (4.12), we have:

$$\tau^*[\mathbf{u}^\delta T_{i_1}] = \mathbf{x}^{S_{i_1, j_1} + \delta} [\tilde{c}_{i_1, j_1} + \Omega(\mathbf{x}, \mathbf{y}, z, \mathbf{w})]$$

where $\tilde{\delta} = \delta \mathbf{A}_1$, the function $\tilde{c}_{i_1, j_1} = c_{i_1, j_1}(\mathbf{y} - \gamma)^{\delta \Lambda}$ is a unit and $\Omega(\mathbf{0}, \mathbf{y}, z, \mathbf{w}) \equiv 0$. Since $\mathbf{x}^{S_{i_1, j_1} + \delta}$ is not a first integral of $\tilde{\theta}$ (which is clear from that fact that α_1 doesn't belong to the span of the rows of \mathbf{A}_1), we conclude that $\nu(q, \theta_r, \mathcal{R}_r) = 0 < \nu$. (Must talk about the δ).

Case 2.3: [The pull-back of \mathbf{u}^β is the generator of \mathcal{J}^* i.e. $S_0 = \min\{S_\nu, S_0, S_{i,j}\}$ and $S_\nu > S_0$, $S_{i,j} > S_0$] In this case, we recall that there exists i_0 such that

$$\mathbf{x}^{S_0} c_{i_0,0} = \mathbf{x}^{S_0} w_2^\epsilon W$$

where W is a unit. We consider two cases depending on ϵ :

Case 2.3a, $\epsilon = 1$: Then

$$\tau^* T_{i_0} = \mathbf{x}^{S_0} [w_2 W + \Omega(\mathbf{x}, \mathbf{y}, z, \mathbf{w})]$$

where W is a unit and $\Omega(\mathbf{0}, \mathbf{y}, z, \mathbf{w}) \equiv 0$. Since the vector-field ∂_{w_2} clearly belongs to $\theta_r \cdot \mathcal{O}_q$, we conclude that $\nu(q, \theta_r, \mathcal{R}_r) \leq 1 < \nu$.

Case 2.3b, $\epsilon = 0$: In this case, notice that the monomial $\mathbf{u}^{\beta + \delta}$ is **not** a first integral of $\theta_r \cdot \mathcal{O}_p$. Thus, Lemma 4.8 guarantees that the total transform $\mathbf{u}^{\beta + \delta} = \mathbf{x}^{S_0 + \delta} \tilde{W}$, where $\tilde{W} = W(\mathbf{y} - \gamma)^{\delta \Lambda}$ is a unit, satisfies one of the following:

- Either $\mathbf{x}^{S_0+\tilde{\delta}}$ is **not** a first integral of $\tilde{\theta}.\mathcal{O}_q$, which implies that

$$\tau^*[\mathbf{u}^\delta T_{i_0}] = \mathbf{x}^{S_0+\tilde{\delta}} U$$

for some unit U . We conclude that $\nu(q, \theta_r, \mathcal{R}_r) = 0 < \nu$;

- Or, there exists a regular vector-field $\partial_{y_i} \in \tilde{\theta}.\mathcal{O}_q$ such that $\partial_{y_i} \widetilde{W}$ is a unit. In particular:

$$\tau^*[\mathbf{u}^\delta T_{i_0}] = \mathbf{x}^{S_0+\tilde{\delta}} \widetilde{W}(0) + \mathbf{x}^{S_0+\tilde{\delta}} [\widetilde{W} - \widetilde{W}(0) + \Omega(\mathbf{x}, \mathbf{y}, z, \mathbf{w})]$$

where $\Omega(\mathbf{0}, \mathbf{y}, z, \mathbf{w}) \equiv 0$ and the monomial $\mathbf{x}^{S_0+\tilde{\delta}} \widetilde{W}(0)$ is a first integral of $\tilde{\theta}.\mathcal{O}_q$. We conclude that $\nu(q, \theta_r, \mathcal{R}_r) \leq 1 < \nu$.

□

4.5 Proof of Theorem 3.7

We suppose that we are in the hypothesis of Theorem 3.7. If $\nu = \nu(p, \theta, \mathcal{R})$ is infinite, then we apply Lemma 4.1 to obtain a θ -admissible collection of local blowings-up $\tau_i : (M_i, \theta_i, \mathcal{R}_i, E_i) \rightarrow (M, \theta, \mathcal{R}, E)$ such that, for every point q_i in the pre-image of p , the invariant $\nu(q_i, \theta_i, \mathcal{R}_i)$ is finite.

So, let us assume that the invariant $\nu := \nu(p, \theta, \mathcal{R})$ is finite. Then, by proposition 4.4, there exists a θ -admissible collection of local blowings-up $\sigma_i : (M_i, \theta_i, \mathcal{R}_i, E_i) \rightarrow (M, \theta, \mathcal{R}, E)$ such that: at every point q_i in the pre-image of p by σ_i , the d -foliated sub-ring sheaf $(M_i, \theta_i, \mathcal{R}_i, E_i)$ satisfies the Prepared Normal Form at q_i with $\nu(q_i, \theta_i, \mathcal{R}_i) \leq \nu$.

Now, by Lemma 4.9 and compacity of the pre-image of p , there exists a θ -admissible collection of local blowings-up $\sigma_{i,j} : (M_{i,j}, \theta_{i,j}, \mathcal{R}_{i,j}, E_{i,j}) \rightarrow (M_i, \theta_i, \mathcal{R}_i, E_i)$ such that: at every point $q_{i,j}$ in the pre-image of p by $\sigma_i \circ \sigma_{i,j}$, the invariant $\nu(q_{i,j}, \theta_{i,j}, \mathcal{R}_{i,j})$ is strictly smaller than the initial invariant ν . Thus, taking the finite collection of morphisms $\tau_{i,j} := \sigma_i \circ \sigma_{i,j}$, we obtain the necessary sequence of local blowings-up.

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